Chapter 4

Renormalisation Group

Previously, our analysis of the Ginzburg-Landau Hamiltonian revealed a formal breakdown of mean-field theory in dimensions below some upper critical dimension. Although the integrity of mean-field theory is sometimes extended by resolution limitations in experiment, the breakdown of mean-field theory is often associated with the appearance of qualitatively new critical behaviour. In the previous section, we saw that a simple scaling hypothesis can lead to useful insight into critical behaviour below the upper critical dimension. However, to complement the ideas of scaling, a formal theoretical approach to the analysis of the Ginzburg-Landau Hamiltonian is required. In this section we will introduce a general scheme which allows one to explore beyond the realms of mean-field theory. Yet the method, known as the Renormalisation Group, is not exact nor completely controlled. Instead, it should be regarded as largely conceptual — i.e. its application, which relies fundamentally only on scaling, can be tailored to the particular application at hand.

4.1 Conceptual Approach

The success of the scaling theory in correctly predicting various exponent identities strongly supports the contention that close to the critical point the correlation length ξ is the only important length scale, and that the microscopic lengths are irrelevant. The critical behaviour is governed by fluctuations that are statistically self-similar up to the scale ξ. Can this self-similarity be used to develop a theory of critical phenomena below the upper critical dimension? Kadanoff[1] suggested taking advantage of the self-similarity to gradually eliminate the correlated degrees of freedom at length scales $x \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at length scale ξ. This is achieved through a procedure known as the **Renormalisation Group** (RG), whose

[1] Leo Kadanoff: recipient of the 1999 National Medal of Science and the 1998 Lars Onsager Prize "for his numerous and profound contributions to statistical physics, including the introduction of the concepts of universality and block spin scaling that are central to the modern understanding of the critical phenomena".

conceptual foundation is outlined below:

1. **Coarse-Grain**: The first step of the RG is to decrease the resolution by changing the minimum length scale from the microscopic scale a to ba where $b > 1$. This is achieved by integrating out fluctuations of the fields $\mathbf{m}$ which occur on length scales finer than ba. The result is a renormalisation of the Hamiltonian βH which leads to an effective Hamiltonian expressed in terms of a 'coarse-grained' magnetisation field

$$\bar{\mathbf{m}}(\mathbf{x}) = \frac{1}{(ba)^d} \int_{\text{Cell}} d\mathbf{y}\ \mathbf{m}(\mathbf{y}),$$

   where the integral runs over a cell of size $(ba)^d$ centred on $\mathbf{x}$.

2. **Rescale**: Due to the change in resolution, the coarse-grained "picture" is grainier than the original. The original resolution a can be restored by decreasing all length scales by a factor b, i.e. defining

$$\mathbf{x}' = \frac{\mathbf{x}}{b}.$$

   Thus, at each position $\mathbf{x}'$ we have defined an average moment $\bar{\mathbf{m}}(\mathbf{x}')$.

3. **Renormalise**: The relative size of the fluctuations of the rescaled magnetisation profile is in general different from the original, i.e. there is a change in contrast between the pictures. This can be remedied by introducing a factor ζ, and defining a renormalised magnetisation

$$\mathbf{m}'(\mathbf{x}') = \frac{1}{\zeta}\bar{\mathbf{m}}(\mathbf{x}').$$

   The choice of ζ will be discussed later.

By following these steps, for each configuration $\mathbf{m}(\mathbf{x})$ one generates a renormalised configuration $\mathbf{m}'(\mathbf{x}')$. It can be regarded as a mapping of one set of random variables to another, and can be used to construct the probability distribution. Kadanoff's insight was to realise that since, on length scales less than ξ, the renormalised configurations are statistically similar to the original ones, they must be distributed by a Hamiltonian that is also close to the original. In particular, if the original Hamiltonian βH is at a critical point, $t = h = 0$, the new $\beta H'$ is also at criticality since no new length scale is generated in the renormalisation procedure, i.e. $t' = h' = 0$.

However, if the Hamiltonian is originally off criticality, then the renormalisation takes us further away from criticality because $\xi' = \xi/b$ is smaller. The next assumption is that since any transformation only involves changes at the shortest length scales it can not produce singularities. The renormalised parameters must be *analytic* functions, and hence expandable as

$$\begin{cases} t(b;t,h) = A(b)t + B(b)h + O(t^2, h^2, th), \\ h(b;t,h) = C(b)t + D(b)h + O(t^2, h^2, th). \end{cases}$$

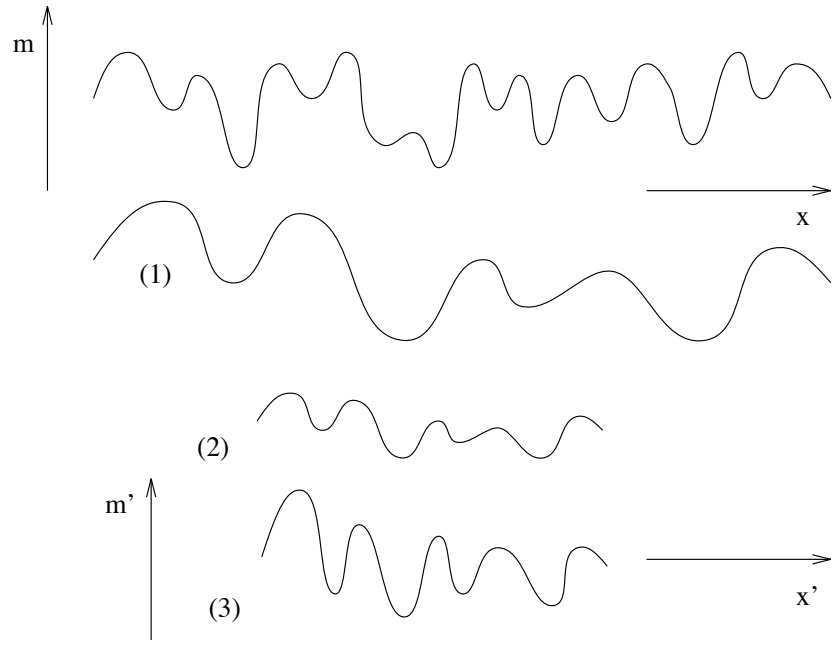


Figure 4.1: Schematic diagram showing the three steps of Kadanoff's renormalisation procedure.

However, the known behaviour at $t = h = 0$ rules out a constant term in the expansion, and to prevent a spontaneously broken symmetry we further require $C(b) = B(b) = 0$. Finally, commutativity $A(b_1 \times b_2) = A(b_1) \times A(b_2)$ implies $A(b) = b^{y_t}$ and $D(b) = b^{y_h}$. So, to lowest order

$$\begin{cases} t(b) = b^{y_t} t, \\ h(b) = b^{y_h} h. \end{cases} \tag{4.1}$$

where y_t, $y_h > 0$ (to ensure that ξ diminishes under the RG procedure). As a consequence:

1. **The free energy**: Since the statistical Boltzmann weight of the new configuration, $\exp[\beta H'[\mathbf{m}']]$ is obtained by summing the weights $\exp[\beta H[\mathbf{m}]]$ of old ones, the partition function is preserved

$$\mathcal{Z} = \int D\mathbf{m}\ e^{-\beta H[\mathbf{m}]} = \int D\mathbf{m}'\ e^{-\beta H'[\mathbf{m}']} = \mathcal{Z}'.$$

From this it follows that the free energies density takes the form

$$f(t,h) = -\frac{\ln \mathcal{Z}}{V} = -\frac{\ln \mathcal{Z}'}{V' b^d} = b^{-d} f(t(b), h(b)) = b^{-d} f(b^{y_t} t, b^{y_h} h), \tag{4.2}$$

where we have assumed that the two free energies are obtained from the *same Hamiltonian* in which only the parameters t and h have changed according to Eq. (4.1). Eq. (4.2) describes a *homogeneous* function of t and h. This is made apparent by

choosing a rescaling factor b such that $b^{y_t}t$ is a constant, say unity, i.e. $b = t^{-1/y_t}$, and

$$\boxed{f(t,h) = t^{d/y_t} f(1, h/t^{y_h/y_t}) \equiv t^{d/y_t} g_f(h/t^{y_h/y_t})}$$

We have thus recovered the scaling form of Eq. (3.2) and can identify the exponents

$$\boxed{2 - \alpha = d/y_t, \qquad \Delta = y_h/y_t} \tag{4.3}$$

So if y_t and y_h are known we can generate all critical exponents.

2. **Correlation Length**: All length scales are reduced by a factor of b during the RG transformation. This is also true of the correlation length $\xi' = \xi/b$ implying

$$\boxed{\xi(t,h) = b\, \xi(b^{y_t}t, b^{y_h}h) = t^{-1/y_t}\xi(1, h/t^{y_h/y_t}) = t^{-1/y_t} g_\xi(h/t^{y_t/y_h})}$$

This identifies $\nu = 1/y_t$ and using Eq. (4.3), the hyperscaling identity $2 - \alpha = d\nu$ is recovered.

3. **Magnetisation**: From the homogeneous form of the free energy we can obtain other bulk quantities such as magnetisation. Alternatively, from the RG results for $\mathcal{Z}$, V, and h we conclude

$$\boxed{m(t,h) = \frac{1}{V}\frac{\partial \ln \mathcal{Z}(t,h)}{\partial h} = \frac{1}{b^d V'}\frac{1}{b^{-y_h}}\frac{\partial \ln Z'(t',h')}{\partial h'} = b^{y_h - d} m(b^{y_t}t, b^{y_h}h)}$$

Choosing $b = t^{-1/y_t}$, we find $m(t,h) = t^{-(y_h-d)/y_t} g_m(h/t^{y_h/y_t})$ which implies that $\beta = (y_h - d)/y_t$ and $\Delta = y_h/y_t$ as before.

It is therefore apparent that quite generally, a quantity X will have a homogeneous form

$$X(t,h) = b^{y_X} X(b^{y_t}t, b^{y_h}h) = t^{-y_X/y_t} g_X(h/t^{y_h/y_t}). \tag{4.4}$$

In general, for any conjugate pair of variables contributing a term $\int d\mathbf{x}\ \mathbf{F}\cdot\mathbf{X}$ to the Hamiltonian (e.g. $\mathbf{m}\cdot\mathbf{h}$), the **scaling dimensions** are related by $y_X + y_F = d$.

4.2 Formal Approach

In the previous section we found that all critical properties can be abstracted from a scaling relation. Though conceptually appealing, it is not yet clear how such a procedure can be formally implemented. In particular, why should the form of the two Hamiltonians be identical, and why are the two parameters t and h sufficient to describe the transition? In this section we outline a more formal procedure for identifying the effects of the dilation operation on the Hamiltonian. The various steps of the program are as follows:

1. Start with the most general Hamiltonian allowed by symmetry. For example, in the presence of rotational symmetry,

$$\beta H[\mathbf{m}] = \int d\mathbf{x} \left[\frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + v\mathbf{m}^6 + \cdots + \frac{K}{2}(\nabla \mathbf{m})^2 + \frac{L}{2}(\nabla^2 \mathbf{m})^2 + \cdots \right]. \quad (4.5)$$

2. Apply the three steps of the renormalisation in configuration space: (i) Coarse grain by b; (ii) rescale, $\mathbf{x}' = \mathbf{x}/b$; and (iii) renormalise, $\mathbf{m}' = \mathbf{m}/\zeta$. This defines a change of variables

$$\mathbf{m}'(\mathbf{x}') = \frac{1}{\zeta b^d} \int_{\text{Cell centred at } b\mathbf{x}'} d\mathbf{x}\ \mathbf{m}(\mathbf{x}).$$

Given the Boltzmann weight $\exp[-\beta H[\mathbf{m}(\mathbf{x})]]$ of the original configurations, we can use the change of variables above to construct the corresponding weight $\exp[-\beta H'[\mathbf{m}'(\mathbf{x}')]]$ of the new configurations. Naturally this is the most difficult step in the program.

3. Since rotational symmetry is preserved by the RG procedure, the rescaled Hamiltonian must also be described by a point in parameter space,

$$\beta H'[\mathbf{m}'] = \int d\mathbf{x}' \left[\frac{t'}{2}\mathbf{m}'^2 + u'\mathbf{m}'^4 + v'\mathbf{m}'^6 + \cdots + \frac{K'}{2}(\nabla \mathbf{m}')^2 + \frac{L'}{2}(\nabla^2 \mathbf{m}')^2 + \cdots \right].$$

The renormalised coefficients are functions of the original ones, i.e. $t' = t(b; t, u, \cdots)$; $u' = u(b; t, u, \cdots)$, etc., defining a mapping $\mathbf{S}' \mapsto \mathbf{R}_b\mathbf{S}$ in parameter space. In general such a mapping is non-linear.

4. The operation $\mathbf{R}_b$ describes the effects of dilation on the Hamiltonian of the system. Hamiltonians that describe statistically self-similar configurations must thus correspond to **fixed points** $\mathbf{S}^*$ such that $\mathbf{R}_b\mathbf{S}^* = \mathbf{S}^*$. Since the correlation length, a function of Hamiltonian parameters, is reduced by b under the RG operation (i.e. $\xi(\mathbf{S}) = b\ \xi(\mathbf{R}_b\mathbf{S})$), the correlation length at a fixed point must be zero or infinity. Fixed points with $\xi^* = 0$ describe independent fluctuations at each point and correspond to complete disorder (infinite temperature), or complete order (zero temperature). Fixed points with $\xi^* = \infty$ describe the critical point ($T = T_c$).

5. Eq. (4.1) represents a simplified case in which the parameter space is two-dimensional. The point $t = h = 0$ is a fixed point, and the lowest order terms in these equations describe the behaviour in the neighbourhood of the fixed point. In general, we can study the stability of a fixed point by *linearising* the **recursion relations** in its vicinity: under RG, a point $\mathbf{S}^* + \delta\mathbf{S}$ is transformed to

$$S_i^* + \delta S_i' = S_i^* + \sum_j [\mathbf{R}_b]_{ij}\, \delta S_j + \cdots, \qquad (\mathbf{R}_b)_{ij} \equiv \frac{\partial S_i'}{\partial S_j}\Big|_{\mathbf{S}^*}.$$

Because of the semi (i.e. irreversible)-group property we have

$$\mathbf{R}_b\mathbf{R}_{b'}\mathbf{O}_i = \lambda_i(b)\lambda_i(b')\mathbf{O}_i = \mathbf{R}_{bb'}\mathbf{O}_i = \lambda_i(bb')\mathbf{O}_i,$$

where $\mathbf{O}_i$ denote the eigenvectors of $[\mathbf{R}_b]_{ij}$ with the eigenvalues $\lambda_i(b)$. Together with the condition $\lambda_i(b = 1) = 1$, the equation above implies $\lambda_i(b) = b^{y_i}$.

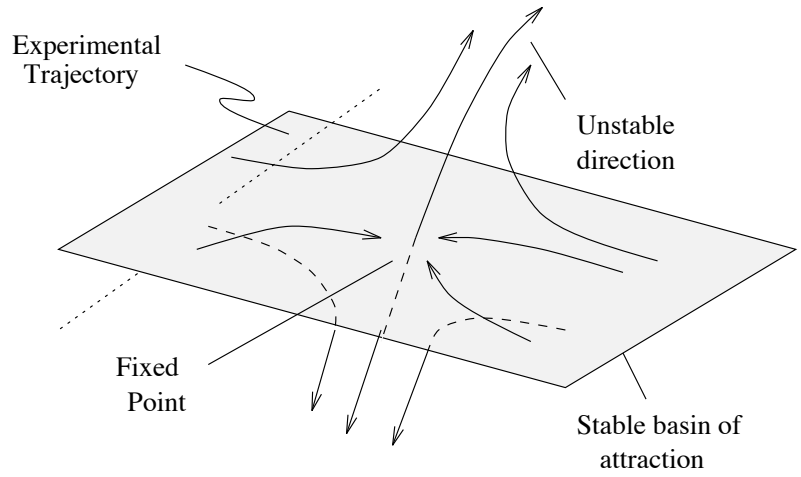


Figure 4.2: Schematic representation of RG flows in a high-dimensional space. Fixed-points describing ordered or disordered phases generally have basins of attraction with dimensions equal to that of the space of potentials. Fixed points describing critical points have a basin of attraction of lower dimensionality.

The vectors $\mathbf{O}_i$ are called **scaling directions** associated with the fixed point S^*, and y_i are the corresponding **anomalous dimensions**. Any Hamiltonian in the vicinity of the fixed point can be described by a set of parameters $\mathbf{S} = \mathbf{S}^* + \sum_i g_i \mathbf{O}_i$. The renormalised Hamiltonian has the interaction parameters $\mathbf{S}' = \mathbf{S}^* + \sum_i g_i b^{y_i} \mathbf{O}_i$.

If $y_i > 0$, g_i *increases* under scaling, and O_i is a **relevant operator**.

If $y_i < 0$, g_i *decreases* under scaling, and O_i is a **irrelevant operator**.

If $y_i = 0$, O_i is a **marginal operator**, and higher order terms are necessary to track the behaviour.

The subspace spanned by the irrelevant directions is call the **basin of attraction** of the fixed point $\mathbf{S}^*$. Since ξ always decreases under RG ($\xi' = \xi/b$), and $\xi(\mathbf{S}^*) = \infty$, ξ is also infinite for any point on the basin of attraction of $\mathbf{S}^*$. The surface defines the phase transition — it is equivalent to varying β (i.e. the temperature) at different values of the parameters and eventually meeting the surface.

In fact for a general point in the vicinity of S^*, the correlation length satisfies the relation

$$\xi(g_1, g_2, \cdots) = b\xi(b^{y_1} g_1, b^{y_2} g_2, \cdots). \tag{4.6}$$

For sufficiently large b all the irrelevant operators scale to zero. The leading singularities of ξ are then determined by the remaining set of *relevant* operators. In particular, if the operators are indexed in order of decreasing dimensions, we can choose b such that $b^{y_1} g_1 = 1$. In this case Eq. (4.6) implies

$$\xi(g_1, g_2, \cdots) = g_1^{-1/y_1} f(g_2/g_1^{y_2/y_1}, \cdots).$$

We have thus obtained an exponent $\nu_1 = 1/y_1$ for the divergence of ξ, and a generalised set of gap exponents $\Delta_\alpha = y_\alpha/y_1$ associated with g_α.

Let us imagine that the fixed point $\mathbf{S}^*$ describes the critical point of the magnet in Eq. (4.5) at zero magnetic field. As the temperature, or some other control parameter, is changed, the coefficients of the Hamiltonian are altered, and the point S follows a different trajectory in parameter space under renormalisation (see Fig. 4.2). Except for a single point (at the critical temperature) the magnet has a finite correlation length. This can be achieved if the experimental trajectory of the unrenormalised parameters $\mathbf{S}$ intersects the basin of attraction of $\mathbf{S}^*$ only at one point. To achieve this the basin must have co-dimension one, i.e. the fixed point S^* must have one and only one relevant operator.

This provides an explanation of **universality** in that the very many microscopic details of the system make up a huge space of irrelevant operators comprising the basin of attraction. In the presence of a magnetic field, two system parameters must be adjusted to reach the critical point, ($T = T_c$ and $h = 0$). Thus the magnetic field corresponds to an additional relevant operator of $\mathbf{S}^*$. *In general, for fixed points describing second-order critical points, there are two relevant parameters: the temperature and the field conjugate to the order parameter* (for the magnet it is the magnetic field).

Although the formal procedure outlined in this section is quite rigorous, it suffers from some quite obvious shortcomings: how do we actually implement the RG transformations analytically? There are an infinite number of interactions allowed by symmetry, and hence the space of parameters of $\mathbf{S}$ is inconveniently large. How do we know *a priori* that there are fixed points for the RG transformation; that $\mathbf{R}_b$ can be linearised; that relevant operators are few; etc? The way forward was presented by Wilson[2] who showed how these steps can be implemented (at least perturbatively) in the Ginzburg-Landau model.

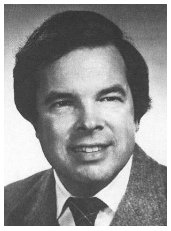
4.3 The Gaussian Model

In this section we will apply the RG approach to study the **Gaussian theory** obtained by retaining only the terms to quadratic order in the Ginzburg-Landau Hamiltonian,

$$\mathcal{Z} = \int D\mathbf{m}(\mathbf{x}) \exp\left\{ - \int d^d\mathbf{x} \left[\frac{t}{2}\mathbf{m}^2 + \frac{K}{2}(\nabla \mathbf{m})^2 - \mathbf{h}\cdot\mathbf{m} \right] \right\}, \tag{4.7}$$

where, as usual, $\mathbf{m}$ represents an n-component vector field. The absence of a term at order m^4 makes the Hamiltonian meaningful only for $t \geq 0$. The singularity at $t = 0$ can be considered as representing the ordered side of the phase transition.

[2] Kenneth G. Wilson, 1936-: Recipient of the 1982 Nobel Prize in Physics, awarded for "discoveries he made in understanding how bulk matter undergoes phase transition, i.e., sudden and profound structural changes resulting from variations in environmental conditions". Wilson's background ranges from elementary particle theory and condensed matter physics (critical phenomena and the Kondo problem) to quantum chemistry and computer science.



4.3.1 Exact Solution

Before turning to the RG analysis, let us first obtain the exact homogeneous form for the free energy density. Being of quadratic form, the Hamiltonian is diagonalised in Fourier space and generates the partition function[3]

$$\mathcal{Z} = \int D\mathbf{m}(\mathbf{q}) e^{-\beta H[\mathbf{m}]}, \qquad \beta H[\mathbf{m}] = \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left(t + K\mathbf{q}^2\right) |\mathbf{m}(\mathbf{q})|^2 - \mathbf{h}\cdot\mathbf{m}(\mathbf{q}=0).$$

Performing the Gaussian integral, and neglecting the constant factor $(2\pi)^{nN/2}$ arising from the Gaussian functional integral, we obtain the free energy density,

$$f(t,h) = -\frac{\ln \mathcal{Z}}{V} = \frac{n}{2}\int \frac{d\mathbf{q}}{(2\pi)^d} \ln(t + K\mathbf{q}^2) - \frac{h^2}{2t}.$$

Although the integral runs over the whole Fourier space $\mathbf{q}$, the important singular contributions originate from long wavelength modes (i.e. those around $\mathbf{q}=0$). To study the non-analytic contributions to f it is convenient to approximate the domain of integration by a "hypersphere" of radius $\Lambda \approx \pi/a$ where a denotes the short-length scale cut-off. The functional form of the integral can be obtained on dimensional grounds by rescaling $\mathbf{q}$ by a factor $\sqrt{t/K}$. Neglecting the upper limit to the integral, and logarithmic factors, the free energy takes the scaling form

$$f_{\text{sing.}}(t,h) = t^{d/2}\left[A + B\frac{h^2}{t^{1+d/2}}\right] \equiv t^{2-\alpha} g_f(h/t^{\Delta}),$$

where A and B represent dimensionless constants.

Thus, matching the points ($h=0$, $t=0^+$) and ($h\to 0$), the singular part of the free energy is described by the exponents

$$\boxed{\alpha_+ = 2 - d/2, \qquad \Delta = (2+d)/4}$$

Since the ordered phase for $t<0$ is not stable, the exponent β is undefined. The susceptibility, $\chi \propto \partial^2 f/\partial h^2 \propto 1/t$, diverges with an exponent $\gamma_+ = 1$.

4.3.2 The Gaussian Model via RG

The RG of the Gaussian model is most conveniently performed in terms of the Fourier modes. The goal is to evaluate the partition function (4.7) indirectly via the three steps of the RG:

[3]Setting $\mathbf{m}(\mathbf{x}) = \int (d\mathbf{q}/(2\pi)^d)\, \mathbf{m}(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}}$, $\mathbf{m}(\mathbf{q}) = \int d\mathbf{x}\, \mathbf{m}(\mathbf{x})\, e^{-i\mathbf{q}\cdot\mathbf{x}}$,

$$\begin{aligned}\int d\mathbf{x}\, \mathbf{m}(\mathbf{x})\cdot\mathbf{m}(\mathbf{x}) &= \int \frac{d\mathbf{q}}{(2\pi)^d)} \int \frac{d\mathbf{q}'}{(2\pi)^d} \mathbf{m}(\mathbf{q})\cdot\mathbf{m}(\mathbf{q}') \overbrace{\int \frac{d\mathbf{x}}{L^d} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}}^{(2\pi)^d\delta(\mathbf{q}+\mathbf{q}')} \\ &= \int \frac{d\mathbf{q}}{(2\pi)^d}\, \mathbf{m}(\mathbf{q})\cdot\mathbf{m}(-\mathbf{q}) = \int \frac{d\mathbf{q}}{(2\pi)^d}\, |\mathbf{m}(\mathbf{q})|^2,\end{aligned}$$

where we have used in the identity $\mathbf{m}^*(\mathbf{q}) = \mathbf{m}(-\mathbf{q})$.

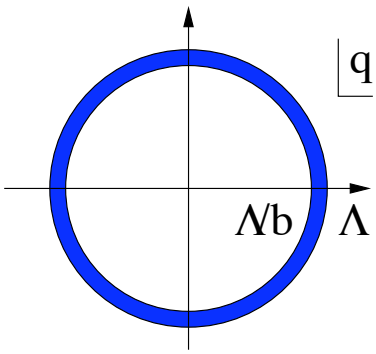


Figure 4.3: Diagram showing the shell in Fourier space that is integrated out in the renormalisation procedure.

1. **Coarse-Grain:** The first step involves the elimination of fluctuations at scales $a < |\mathbf{x}| < ba$. In spirit, it is similar to removing Fourier modes with wavenumbers $\Lambda/b < |\mathbf{q}| < \Lambda$ (see Fig. 4.3.1). We thus separate the fields into slowly and rapidly varying functions, $\mathbf{m}(\mathbf{q}) = \mathbf{m}_>(\mathbf{q}) + \mathbf{m}_<(\mathbf{q})$, with
$$\mathbf{m}(\mathbf{q}) = \begin{cases} \mathbf{m}_<(\mathbf{q}) & 0 < |\mathbf{q}| < \Lambda/b, \\ \mathbf{m}_>(\mathbf{q}) & \Lambda/b < |\mathbf{q}| < \Lambda. \end{cases}$$
The partition function can be re-expressed in the form
$$\mathcal{Z} = \int D\mathbf{m}_<(\mathbf{q}) \int D\mathbf{m}_>(\mathbf{q}) e^{-\beta H[\mathbf{m}_<,\mathbf{m}_>]}.$$
Since the two sets of modes are decoupled in the Gaussian model, the integration is straightforward, and gives
$$\mathcal{Z} = \mathcal{Z}_> \int D\mathbf{m}_<(\mathbf{q}) \exp\left[- \int_0^{\Lambda/b} \frac{d\mathbf{q}}{(2\pi)^d} \left(\frac{t + K\mathbf{q}^2}{2} \right) |\mathbf{m}_<(\mathbf{q})|^2 + \mathbf{h} \cdot \mathbf{m}_<(0) \right],$$
where $\mathcal{Z}_> = \exp[-(nV/2) \int_{\Lambda/b}^{\Lambda} (d\mathbf{q}/(2\pi)^d) \ln(t + k\mathbf{q}^2)]$.

2. **Rescale:** The partition function for the modes $\mathbf{m}_<(\mathbf{q})$ is similar to the original, except that the upper cut-off has decreased to Λ/b, reflecting the coarse-graining in resolution. The rescaling, $\mathbf{x}' = \mathbf{x}/b$ in real space, is equivalent to $\mathbf{q}' = b\mathbf{q}$ in momentum space, and restores the cut-off to the original value.

3. **Renormalise:** The final step of the RG involves the renormalisation of magnetisation field, $\mathbf{m}'(\mathbf{x}') = \mathbf{m}_<(\mathbf{x}')/\zeta$. Alternatively, we can renormalise the Fourier modes according to $\mathbf{m}'(\mathbf{q}') = \mathbf{m}_<(\mathbf{q}')/z$, resulting in
$$\mathcal{Z} = \mathcal{Z}_> \int D\mathbf{m}'(\mathbf{q}') e^{-\beta H'[\mathbf{m}'(\mathbf{q}')]},$$
$$\beta H' = \int_0^{\Lambda} \frac{d\mathbf{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{t + Kb^{-2}\mathbf{q}'^2}{2} \right) |\mathbf{m}'(\mathbf{q}')|^2 - z\mathbf{h} \cdot \mathbf{m}'(0).$$

The constant factor change from the Jacobian can be neglected in favour of the singular contribution from the exponent.

This procedure has transformed from a set of parameters $\mathbf{S} = \{K, t, h\}$ to a new set

$$\mathbf{S}' = \begin{cases} K' = Kb^{-d-2}z^2, \\ t' = tb^{-d}z^2, \\ h' = hz. \end{cases}$$

(Note that in general, such transformations can and often will lead to the appearance of new terms absent in the original Hamiltonian.) The singular point $t = h = 0$ is mapped onto itself as expected. To make the fluctuations scale invariant at this point, we must ensure that the remaining parameter in the Hamiltonian, K, stays fixed. This is achieved by the choice $z = b^{1+d/2}$ which implies that

$$\begin{cases} t' = b^2 t & y_t = 2, \\ h' = b^{1+d/2}h & y_h = 1 + d/2. \end{cases}$$

For the fixed point $t = t'$, K becomes weaker and the spins become uncorrelated — the high temperature phase.

From these equations, we can predict the scaling of the Free energy

$$\begin{aligned} f_{\text{sing.}}(t, h) &= b^{-d} f_{\text{sing.}}(b^2 t, b^{1+d/2}h), \qquad b^2 t = 1, \\ &= t^{d/2} g_f(h/t^{1/2+d/4}). \end{aligned}$$

This implies exponents: $2 - \alpha = d/2$, $\Delta = y_h/y_t = 1/2 + d/4$, and $\nu = 1/y_t = 1/2$. Comparing with the results from the exact solution we can can confirm the validity of the RG.

At the fixed point ($t = h = 0$) the Hamiltonian must be scale invariant. This allows us to fix the value of the renormalisation parameter ζ. By dimensional analysis $\mathbf{x} = b\mathbf{x}'$, $\mathbf{m}(\mathbf{x}) = \zeta \mathbf{m}'(\mathbf{x}')$ and

$$\beta H^* = \frac{K}{2} b^{d-2} \zeta^2 \int d\mathbf{x}' \; (\nabla \mathbf{m}')^2, \qquad \zeta = b^{1-d/2}.$$

Therefore, for small perturbations

$$\beta H^* + u_p \int d\mathbf{x} \; |\mathbf{m}(\mathbf{x})|^p \to \beta H^* + u_p b^d \zeta^p \int d\mathbf{x}' \; |\mathbf{m}'(\mathbf{x}')|^p.$$

Thus, in general $u_p \mapsto u_p' = b^d b^{p - pd/2} = b^{y_p} u_p$, where $y_p = p - d(p/2 - 1)$, in agreement with our earlier findings that $y_1 \equiv y_h = 1 + d/2$ and $y_2 \equiv y_t = 2$. For the Ginzburg-Landau Hamiltonian, the quartic term scales with an exponent $y_4 = 4 - d$ and is therefore relevant for $d < 4$ and irrelevant for $d > 4$. Sixth order perturbations scale with an exponent $y_6 = 6 - 2d$ and is therefore irrelevant for $d > 3$.

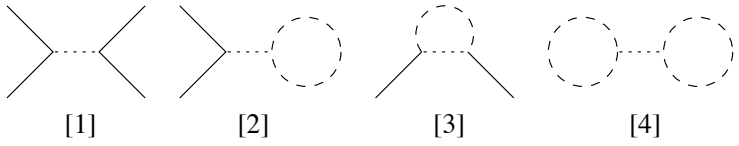


Figure 4.4: Diagrammatic representation of the correlators appearing in the text.

4.4 Wilson's Perturbative Renormalisation Group

In this section we will assess the extent to which the higher order terms in the Ginzburg-Landau expansion can be treated as a perturbation of the Gaussian model. Our method will be to combine the momentum space RG with a perturbative treatment of the Hamiltonian.

Since the unperturbed part of the Hamiltonian is diagonal in Fourier space, it is convenient to switch to that representation and re-express

$$\beta H[\mathbf{m}] = \overbrace{\int d\mathbf{x}\left[\frac{t}{2}\mathbf{m}^2+\frac{K}{2}(\nabla\mathbf{m})^2\right]}^{\beta H_0} + \overbrace{u\int d\mathbf{x}\mathbf{m}^4}^{U},$$

as

$$\begin{aligned}\beta H_0 &= \int \frac{d\mathbf{q}}{(2\pi)^d}\frac{1}{2}\left(t+K\mathbf{q}^2\right)|\mathbf{m}(\mathbf{q})|^2,\\ U &= u\int\frac{d\mathbf{q}_1}{(2\pi)^d}\int\frac{d\mathbf{q}_2}{(2\pi)^d}\int\frac{d\mathbf{q}_3}{(2\pi)^d}\mathbf{m}(\mathbf{q}_1)\cdot\mathbf{m}(\mathbf{q}_2)\;\mathbf{m}(\mathbf{q}_3)\cdot\mathbf{m}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3).\end{aligned}$$

To implement the perturbative RG we proceed, as before, in three steps

1. **Coarse-Grain**: Subdividing the fluctuations into two components $\mathbf{m}(\mathbf{q}) = \mathbf{m}_<(\mathbf{q}) + \mathbf{m}_>(\mathbf{q})$ the contribution to the unperturbed (Gaussian) part of the Hamiltonian is separable while the perturbation mixes the terms. Integrating, we obtain

$$\begin{aligned}\mathcal{Z} &= \mathcal{Z}_0^> \int D\mathbf{m}_< e^{-\beta H_0[\mathbf{m}_<]}\overbrace{\frac{1}{\mathcal{Z}_0^>}\int D\mathbf{m}_> e^{-\beta H_0[\mathbf{m}_>]-U[\mathbf{m}_<,\mathbf{m}_>]}}^{\left\langle e^{-U[\mathbf{m}_<,\mathbf{m}_>]}\right\rangle_{\mathbf{m}_>}}\\ &= \mathcal{Z}_0^> \int D\mathbf{m}_< e^{-\beta H_0[\mathbf{m}_<]+\ln\left\langle e^{-U[\mathbf{m}_<,\mathbf{m}_>]}\right\rangle_{\mathbf{m}_>}},\end{aligned}$$

   where $\mathcal{Z}_0^>$ denotes the contribution to the Gaussian (unperturbed) partition function arising from $\mathbf{m}_>$.

   In general, the renormalisation of the Hamiltonian would call for the expansion

$$\ln\left\langle e^{-U}\right\rangle = -\left\langle U\right\rangle + \frac{1}{2}\left(\left\langle U^2\right\rangle - \left\langle U\right\rangle^2\right) + \cdots + \frac{(-1)^\ell}{\ell!}\left\langle U^\ell\right\rangle_c + \cdots,$$

where $\langle U^\ell \rangle_c$ denotes the ℓth cummulant. However, for simplicity, we will stop here at leading order in the perturbation from which we obtain

$$\beta H[\mathbf{m}_<] = \beta H_0[\mathbf{m}_<] - \ln[\mathcal{Z}_0^>] + \langle U \rangle_{\mathbf{m}_>} + O(u^2).$$

Only terms which are of an even order in $\mathbf{m}_>$ contribute to the average $\langle U \rangle_{\mathbf{m}_>}$. In particular, we will require averages of the form

$$\begin{aligned}
C_1(\{\mathbf{q}_i\}) &= \langle \mathbf{m}_<(\mathbf{q}_1)\cdot\mathbf{m}_<(\mathbf{q}_2)\ \mathbf{m}_<(\mathbf{q}_3)\cdot\mathbf{m}_<(\mathbf{q}_4)\rangle_{\mathbf{m}_>}\,,\\
C_2(\{\mathbf{q}_i\}) &= \langle \mathbf{m}_>(\mathbf{q}_1)\cdot\mathbf{m}_>(\mathbf{q}_2)\ \mathbf{m}_<(\mathbf{q}_3)\cdot\mathbf{m}_<(\mathbf{q}_4)\rangle_{\mathbf{m}_>}\,,\\
C_3(\{\mathbf{q}_i\}) &= \langle \mathbf{m}_>(\mathbf{q}_1)\cdot\mathbf{m}_<(\mathbf{q}_2)\ \mathbf{m}_>(\mathbf{q}_3)\cdot\mathbf{m}_<(\mathbf{q}_4)\rangle_{\mathbf{m}_>}\,,\\
C_4(\{\mathbf{q}_i\}) &= \langle \mathbf{m}_>(\mathbf{q}_1)\cdot\mathbf{m}_>(\mathbf{q}_2)\ \mathbf{m}_>(\mathbf{q}_3)\cdot\mathbf{m}_>(\mathbf{q}_4)\rangle_{\mathbf{m}_>}\,.
\end{aligned}$$

C_1 simply generates $U[\mathbf{m}_<]$ while C_4 gives some constant independent of $\mathbf{m}_>$. The important contributions arise from C_2 and C_3 which can be represented diagrammatically as in Fig. 4.4.

For the unperturbed Hamiltonian the two-point expectation value is equal to

$$\langle m_\alpha(\mathbf{q}) m_\beta(\mathbf{q}')\rangle_0 = \delta_{\alpha\beta}\ (2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')\ G_0(\mathbf{q}), \qquad G_0(\mathbf{q}) = \frac{1}{t + K\mathbf{q}^2},$$

where the subscript zero indicates that the average is with respect to the unperturbed (Gaussian) Hamiltonian.[4] Using the results above we find

$$\begin{aligned}
C_2(\{\mathbf{q}_i\}) &= nG_0(\mathbf{q}_1)\ (2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2)\ \mathbf{m}_<(\mathbf{q}_3)\cdot\mathbf{m}_<(\mathbf{q}_4),\\
C_3(\{\mathbf{q}_i\}) &= G_0(\mathbf{q}_1)\ (2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_3)\ \mathbf{m}_<(\mathbf{q}_2)\cdot\mathbf{m}_<(\mathbf{q}_4).
\end{aligned}$$

[4] In general, the expectation value involving any product of $\vec{m}$'s can be obtain from the identity for Gaussian distributed random variables

$$\boxed{\left\langle \exp\left[\int d\mathbf{x}\ \mathbf{a}(\mathbf{x})\cdot\mathbf{m}(\mathbf{x})\right]\right\rangle_0 = \exp\left[\int d\mathbf{x}\int d\mathbf{x}'\frac{1}{2}\ a_\alpha(\mathbf{x})\ \langle m_\alpha(\mathbf{x}) m_\beta(\mathbf{x}')\rangle_0\ a_\beta(\mathbf{x}')\right]}$$

Expanding both sides in powers of $\{\mathbf{a}(\mathbf{x})\}$ we obtain **Wick's theorem**

$$\left\langle \prod_{i=1}^{\ell} m_{\alpha_i}(x_i)\right\rangle_0 = \begin{cases} 0 & \ell \text{ odd},\\ \text{sum over all pairwide contractions} & \ell \text{ even}.\end{cases}$$

For example

$$\begin{aligned}
\left\langle m_{\alpha_i}(x_i)m_{\alpha_j}(x_j)m_{\alpha_k}(x_k)m_{\alpha_l}(x_l)\right\rangle_0 &= \left\langle m_{\alpha_i}(x_i)m_{\alpha_j}(x_j)\right\rangle_0 \left\langle m_{\alpha_k}(x_k)m_{\alpha_l}(x_l)\right\rangle_0\\
&+ \left\langle m_{\alpha_i}(x_i)m_{\alpha_j}(x_k)\right\rangle_0 \left\langle m_{\alpha_k}(x_j)m_{\alpha_l}(x_l)\right\rangle_0 + \left\langle m_{\alpha_i}(x_k)m_{\alpha_l}(x_l)\right\rangle_0 \left\langle m_{\alpha_k}(x_k)m_{\alpha_l}(x_j)\right\rangle_0.
\end{aligned}$$

Moreover, in the presence of a perturbation U, the expectation value of any operator O can be expressed using the identity

$$\begin{aligned}
\langle O\rangle &= \frac{\int Dm\ O\ e^{-\beta H}}{\int Dm\ e^{-\beta H}} = \frac{\int Dm\ O\ [1-U+U^2/2-\cdots]e^{-\beta H_0}}{\int Dm\ [1-U+U^2/2-\cdots]e^{-\beta H_0}}\\
&= \frac{\mathcal{Z}_0\langle O\rangle_0 - \langle OU\rangle_0 + \langle OU^2/2\rangle_0 - \cdots]}{\mathcal{Z}_0[1-\langle U\rangle_0 + \langle U^2/2\rangle_0 - \cdots]} \equiv \sum_n \frac{(-1)^n}{n!}\langle OU^n\rangle_0^c = \left\langle Oe^{-U}\right\rangle_0^c,
\end{aligned}$$

where the different orders in the expansion define the *connected average* denoted by the superscript c.

Dropping the irrelevant constant terms, C_4 and $\ln \mathcal{Z}_0^>$ we find that no new relevant terms appear in the coarse-grained Hamiltonian $\beta H[\mathbf{m}_<]$, and the coefficients K and u are unrenormalised, while

$$t \mapsto \widetilde{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^d} G_0(\mathbf{q}),$$

the factor of $4(n+2)$ arising from enumerating all permutations.

2. **Rescale**: As usual we set $\mathbf{q}' = b\mathbf{q}$.

3. **Renormalise**: Finally we set $\mathbf{m}' = \mathbf{m}_<(\mathbf{q}')/z$ and obtain

$$\beta H'[\mathbf{m}'] = \int_0^{\Lambda} \frac{d\mathbf{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{\widetilde{t} + Kb^{-2}\mathbf{q}'^2}{2}\right) |\mathbf{m}'(\mathbf{q}')|^2$$
$$+ uz^4 b^{-3d} \int_0^{\Lambda} \frac{d\mathbf{q}_1'}{(2\pi)^d} \int_0^{\Lambda} \frac{d\mathbf{q}_2'}{(2\pi)^d} \int_0^{\Lambda} \frac{d\mathbf{q}_3'}{(2\pi)^d}\, \mathbf{m}'(\mathbf{q}_1') \cdot \mathbf{m}'(\mathbf{q}_2')\; \mathbf{m}'(\mathbf{q}_3') \cdot \mathbf{m}'(-\mathbf{q}_1' - \mathbf{q}_2' - \mathbf{q}_3').$$

The renormalised Hamiltonian is defined by

$$t' = b^{-d} z^2 \widetilde{t}, \qquad K' = b^{-d-2} z^2 K, \qquad u' = b^{-3d} z^4 u.$$

As in the Gaussian model, if we set $z = b^{1+d/2}$ such that $K' = K$, there is a fixed point at $t^* = u^* = 0$. The recursion relations for t and u in the vicinity of this point are given by

$$\begin{aligned} t' \equiv t(b) &= b^2 \left[t + 4u(n+2) \int_{\Lambda/b}^{\Lambda} \frac{d^d\mathbf{q}}{(2\pi)^d} G_0(\mathbf{q}) \right], \\ u' \equiv u(b) &= b^{4-d} u \end{aligned}$$

The recursion relation for u at this order is identical to that obtained by dimensional analysis; but that of t is modified. It is conventional to convert the above discrete recursion relations to continuous differential equations by setting $b = e^{\ell}$. For an infinitesimal $\delta\ell$,

$$\begin{aligned} t(b) &\equiv t(1 + \delta\ell + \cdots) = t + \delta\ell \frac{dt}{d\ell} + O(\delta\ell^2), \\ u(b) &= u + \delta\ell \frac{du}{d\ell} + O(\delta\ell^2). \end{aligned}$$

Expanding the recursion relations, we obtain[5]

$$\begin{aligned} \frac{dt}{d\ell} &= 2t + \frac{4u(n+2)K_d\Lambda^d}{t + K\Lambda^2}, \\ \frac{du}{d\ell} &= (4-d)u, \end{aligned}$$

[5]Here we have made use of the approximation

$$\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^d} G_0(\mathbf{q}) \simeq \left(\Lambda - \frac{\Lambda}{b}\right) \Lambda^{d-1} \frac{S_d}{(2\pi)^d} \frac{1}{K\Lambda^2 + t}$$

and set $\Lambda(1 - e^{-\delta\ell}) \simeq \Lambda\, \delta\ell + \cdots$.

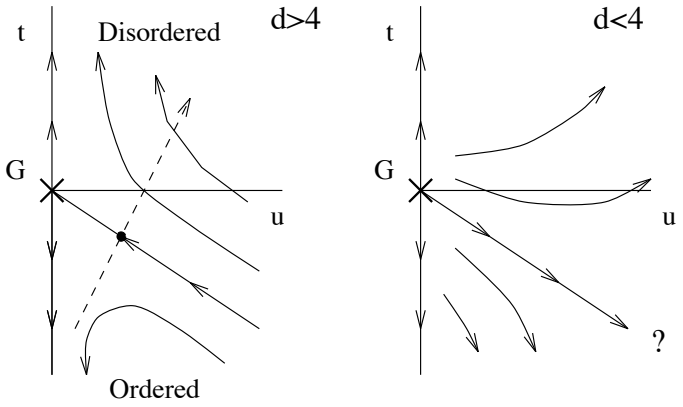


Figure 4.5: Perturbative RG flow of the Ginzburg-Landau model treated within the "one-loop" approximation.

where $K_d \equiv S_d/(2\pi)^d$. Integrated, the second equation gives $u(\ell) = u_0 e^{(4-d)\ell} = u_0 b^{4-d}$.

The recursion relations can be linearised in the vicinity of the fixed point $t^* = u^* = 0$ by setting $t = t^* + \delta t$ and $u = u^* + \delta u$, as

$$\frac{d}{d\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix}2 & 4(n+2)K_d\Lambda^{d-2}/K\\ 0 & 4-d\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}.$$

In the differential form, the eigenvalues of the matrix that enter the recursion relations determine the relevance of the operators. Since the matrix above has zero elements on one side, its eigenvalues are the diagonal elements and, as in the Gaussian model, we can identify $y_t = 2$, and $y_u = 4 - d$. The results at this order are identical to those obtained from dimensional analysis of the Gaussian model. The only difference is in the eigen-directions. The exponent $y_t = 2$ is still associated with $u = 0$, while $y_u = 4 - d$ is actually associated with the direction $t = 4u(n+2)K_d\Lambda^{d-2}/(2-d)K$.

For $d > 4$ the Gaussian fixed point has only one unstable direction associated with y_t. It thus correctly describes the phase transition. For $d < 4$ it has two relevant directions and is unstable. Unfortunately, the recursion relations have no other fixed point at this order and it appears that we have learned little from the perturbative RG. However, since we are dealing with a perturbative series alternating in sign, we can anticipate that the recursion relations at the next order are modified according to

$$\begin{aligned}\frac{dt}{d\ell} &= 2t + \frac{4u(n+2)K_d\Lambda^d}{t + K\Lambda^2} - Au^2,\\ \frac{du}{d\ell} &= (4-d)u - Bu^2,\end{aligned}$$

with A and B both positive. There is now an additional fixed point at $u^* = (4-d)/B$ for $d < 4$. For a systematic perturbation theory we need to keep the parameter u small. Thus the new fixed point can be explored systematically only for $\epsilon = 4 - d$; we are led to consider an expansion in the dimension of space in the vicinity of $d = 4$! For a calculation valid at $O(\epsilon)$ we have to keep track of terms of second order in the recursion relation for

u, but only first order in t. It would thus be unnecessary to calculate the term A in the expression above.

4.5 †The ϵ-Expansion

▷ INFO: It is left as an exercise (see problem set II) to show that expansion to second order ("two-loop") in u leads to the identity

$$B = -\frac{4(n+8)K_d\Lambda^d}{(t+K\Lambda^2)^2}.$$

Thus, in addition to the Gaussian fixed point at $u^* = t^* = 0$, there is now a non-trivial fixed point ($dt/d\ell = du/d\ell = 0$) at

$$\begin{cases} u^* = \dfrac{t^*+K\Lambda^2)^2}{4(n+8)K_d\Lambda^d}\epsilon = \dfrac{K^2}{4(n+8)K_4}\epsilon + O(\epsilon^2), \\ t^* = -\dfrac{2u^*(n+2)K_d\Lambda^d}{t^*+K\Lambda^2} = -\dfrac{(n+2)}{2(n+8)}K\Lambda^2\epsilon + O(\epsilon^2), \end{cases}$$

where only those terms at leading order in $\epsilon = 4-d$ have been retained.

Linearising the recursion relations in the vicinity of the new fixed point we obtain

$$\frac{d}{d\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} 2-\frac{n+2}{n+8}\epsilon & \cdots \\ 0 & -\epsilon\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}.$$

The first eigenvalue is positive controlling the instability of the fixed point

$$\boxed{y_t = 2 - \frac{n+2}{n+8}\epsilon + O(\epsilon^2)}$$

while the second eigenvalue

$$\boxed{y_u = -\epsilon + O(\epsilon^2)}$$

is negative for $d < 4$. The new fixed point thus has co-dimension one and can describe the phase transition in these dimensions. Although the position of the fixed point depends on microscopic parameters such as K and Λ, the final eigenvalues are pure numbers that depend only on n and $d = 4-\epsilon$. These eigenvalues characterise the **universality classes** of rotational symmetry breaking in $d < 4$.

Continuing it is possible to obtain better estimates for critical exponents. However, even at second order, the ϵ-expansion does not make numerically accurate predictions in physical dimensions. Why then should one bother with such calculations? Their great virtue is that they provide a relatively straightforward way of determining what types of universality classes exist. Although the numerical values of the critical exponents change considerably as one moves away from the upper critical dimension, the topology of the flow diagrams does not. Thus one can investigate which interactions will lead to new universality classes and which will not. It is in this sense that the ϵ-expansion is largely responsible for our rather detailed understanding of critical phenomena.

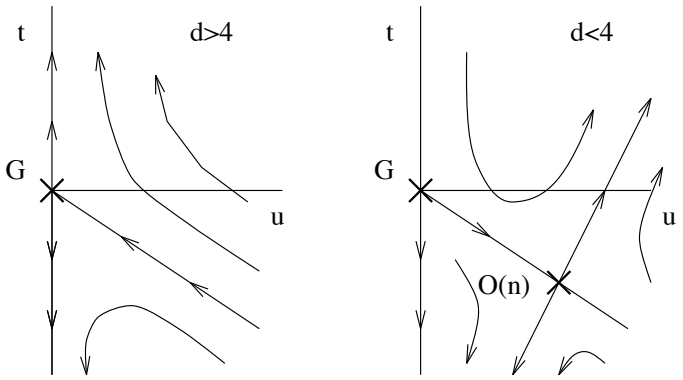


Figure 4.6: Perturbative RG flow of the Ginzburg-Landau model treated within the two-loop approximation.

The perturbative implementation of the RG procedure for the Ginzburg-Landau Hamiltonian was first performed by K. G. Wilson in the early 1970's, while the ϵ-expansion was developed jointly with M. E. Fisher.[6] Wilson was awarded the Nobel prize in 1982. Historical details can be found in his Nobel lecture reprinted in Rev. Mod. Phys. **55**, 583 (1983). This concludes our investigation of the scaling theory and renormalisation group.

---

[6] Michael E. Fisher: recipient of the 1995 Lars Onsager Prize "for his numerous and seminal contributions to statistical mechanics, including but not restricted to the theory of phase transitions and critical phenomena, scaling laws, critical exponents, finite size effects, and the application of the renormalisation group to many of the above problems".