6.2 Answers

1. (a,b) Expanding the expression for the area we obtain the partition function

$$\begin{aligned}\mathcal{Z} &= \int Dh(\mathbf{x})e^{-\beta H[h]}, \\ \beta H &= \beta\sigma A = \beta\sigma \int d^{d-1}\mathbf{x}\left[1 + \frac{1}{2}(\nabla h)^2 + \cdots\right] \\ &= \frac{\beta\sigma}{2}\int (d\mathbf{q})\mathbf{q}^2|h(\mathbf{q})|^2 + \cdots\end{aligned}$$

(c) Making use of the correlator

$$\langle h(\mathbf{q}_1)h(\mathbf{q}_2)\rangle = (2\pi)^{d-1}\delta^{d-1}(\mathbf{q}_1 + \mathbf{q}_2)\frac{1}{\beta\sigma\mathbf{q}_1^2},$$

we obtain the correlator

$$\begin{aligned}\left\langle [h(\mathbf{x}) - h(0)]^2\right\rangle &= \int (d\mathbf{q}_1)(d\mathbf{q}_2)\left(e^{i\mathbf{q}_1\cdot\mathbf{x}} - 1\right)\left(e^{i\mathbf{q}_2\cdot\mathbf{x}} - 1\right)\langle h(\mathbf{q}_1)h(\mathbf{q}_2)\rangle \\ &= \frac{4}{\beta\sigma}\int (d\mathbf{q})\frac{\sin^2(\mathbf{q}\cdot\mathbf{x})}{\mathbf{q}^2}.\end{aligned}$$

By inspection of the integrand, we see that for $d \geq 4$, the integral is dominated by $|\mathbf{q}| \gg 1/|\mathbf{x}|$, and

$$\left\langle [h(\mathbf{x}) - h(0)]^2\right\rangle \sim \text{const}.$$

In three dimensions, the integral is logarithmically divergent and

$$\langle [h(\mathbf{x}) - h(0)]\rangle \sim \frac{1}{\beta\sigma}\ln|\mathbf{x}|.$$

Finally, in two dimensions, the integral is dominated by small $\mathbf{q}$ and

$$\langle [h(\mathbf{x}) - h(0)]\rangle \sim |\mathbf{x}|.$$

This result shows that in dimensions less than 4, a surface constrained only by its tension is unstable due to long-wavelength fluctuations.

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2. Essay question: Refer to Lecture notes

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3. Solving the classical equation of motion $\ddot{r} = -\omega^2 r$, a general solution takes the form

$$r(t) = A\cos(\omega t) + B\sin(\omega t).$$

Applying the boundary conditions $r(0) = 0$ and $r(\bar{t}) = \bar{r}$ we find $A = 0$ and $B = \bar{r}/\sin(\omega\bar{t})$,

$$r_{\rm cl}(t) = \bar{r}\frac{\sin(\omega t)}{\sin(\omega\bar{t})}.$$

From this result we obtain the classical Lagrangian

$$\mathcal{L}[r_{\rm cl}] = \frac{1}{2}m\omega^2\frac{\bar{r}^2}{\sin^2(\omega\bar{t})}\left[\cos^2(\omega t) - \sin^2(\omega t)\right] = \frac{1}{2}m\omega^2\bar{r}^2\frac{\cos(2\omega t)}{\sin^2(\omega\bar{t})}.$$

and the classical action

$$S[r_{\rm cl}] = \int_0^{\bar{t}}\mathcal{L}[r_{\rm cl}]dt = \frac{1}{2}m\omega\bar{r}^2\cot(\omega\bar{t}).$$

Being quadratic, fluctuations around the classical trajectory generate the expansion

$$\mathcal{L}[r_{\rm cl} + \delta r] = \mathcal{L}[r_{\rm cl}] + \mathcal{L}[\delta r].$$

Expanding as a Fourier series,

$$\delta r(t) = \sum_n a_n \sin\left(\frac{\pi n t}{\bar{t}}\right),$$

we find

$$\int_0^{\bar{t}}(\delta r(t))^2 dt = \sum_{nm} a_n a_m \int_0^{\bar{t}}\sin\left(\frac{\pi n t}{\bar{t}}\right)\sin\left(\frac{\pi m t}{\bar{t}}\right)dt = \frac{\bar{t}}{2}\sum_n a_n^2.$$

Similarly

$$\int_0^{\bar{t}}(\delta\dot{r})^2 dt = \frac{\bar{t}}{2}\sum_n\left(\frac{\pi n}{\bar{t}}\right)^2 a_n^2.$$

Altogether, we obtain

$$S[\delta r] = \frac{m\bar{t}}{4}\sum_n a_n^2\left[\left(\frac{\pi n}{\bar{t}}\right)^2 - \omega^2\right].$$

Using the Gaussian integral,

$$\prod_n\int_{-\infty}^{\infty}da_n e^{(i/\hbar)S[\delta r]} = \prod_{n=1}^{\infty}\left(\frac{2\pi\hbar}{im\frac{\bar{t}}{2}\left(\frac{\pi n}{\bar{t}}\right)^2\left[1 - \left(\frac{\omega\bar{t}}{\pi n}\right)^2\right]}\right)^{1/2},$$

we obtain the propagator

$$\mathcal{Z} = J \prod_{n=1}^{\infty} \left(\frac{2\pi\hbar}{im\frac{\bar{t}}{2}\left(\frac{\pi n}{t}\right)^2} \right)^{1/2} \left(2\pi i\hbar \frac{\bar{t}}{m} \right)^{1/2} \left(\frac{m\omega}{2\pi i\hbar \sin(\omega \bar{t})} \right)^{1/2} e^{(i/\hbar)S[r_{\rm cl}]}.$$

In the limit $\omega \to 0$, $S[r_{\rm cl}] = m\bar{r}^2/2\bar{t}$, and

$$\left(\frac{m\omega}{2\pi i\hbar \sin(\omega \bar{t})} \right)^{1/2} \to \left(\frac{m}{2\pi i\hbar \bar{t}} \right)^{1/2},$$

the free particle result. Thus we can deduce that

$$J = \left(\frac{m}{2\pi i\hbar \bar{t}} \right)^{1/2} \prod_{n=1}^{\infty} \left(\frac{im(\bar{t}/2)(\pi n/t)^2}{2\pi\hbar} \right)^{1/2}.$$

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4. To obtain full marks, the answer to the first part of the question should involve an account of spontaneous symmetry breaking in systems with a continuous symmetry. Marks will be given for writing a generic expression for the Ginzburg-Landau free energy describing Goldstone fluctuations; an estimate of the correlation functions in dimensionality $d = 2$, 3 and 4; a definition of the lower critical dimension, and a statement of the Mermin-Wagner theorem. Additional marks will be given for the mention of examples. It is expected that the calculation of the Green function for a point charge in d dimensions can be performed with the use of Gauss' theorem.

   The calculation itself is a simple subset of the first part of the problem. The idea is that, by virtue of solving the technical part of the question, the student can use more time for discussion in the first part of the problem.

   Applying Gauss' theorem as in the notes, the real space representation of the propagator can be found directly,

$$\left\langle [\varphi(\mathbf{x}) - \varphi(0)]^2 \right\rangle = \frac{2}{(2-d)S_d} \frac{1}{\rho_s} \left(x^{2-d} - a^{2-d} \right),$$

   where a is the ultraviolet cut-off. The logarithmic dependence in two-dimensions should be easy to extract. As for the numerical prefactor, if all else fails, it can be deduced from the answer given.

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5. All four topics are book-work. Credit will given for a discussion of concepts rather than a verbatim reworking of the lecture notes. This is particularly true with sections (c) and (d) which are otherwise rather trivial. By virtue of studying past papers, the students should have practised writing a solution to all of these problems.

   (a) Look for a definition of a second-order phase transition, and a brief introduction to the philosophy that lies behind the Ginzburg-Landau theory; a statement of what is meant by Landau theory, and an indication of how it can be used to obtain critical exponents; a mention of how fluctuations lead to a breakdown of mean-field theory; Mermin-Wagner theorem and the definition of the lower critical dimension; Ginzburg criterion and the definition of the upper critical dimension.

(b) It is difficult to imagine that any of the students will attempt this part. If they do, I would look for a general discussion of the importance of the scaling concept in statistical mechanics; how far one can get with the homogeneity assumption; a calculation of some the critical exponent identities; hyperscaling; and how these ideas fit in with the Ginzburg-Landau theory.

(c) I expect that this question will be popular with those students who attempt problem (1). I would expect a brief discussion of the high and low temperature expansion; a derivation of the lower temperature power law decay of spin-spin correlation functions turning over to exponential decay at high temperatures; a classification of topological versus hydrodynamic degrees of freedom and a definition of vortex configurations; an estimate of the condensation temperature; a qualitative discussion of the binding energy of vortex pairs and the unbinding transition.

(d) This is another easy option. For this reason extra credit will be given to students who provide a simple example. In answer to this question, I would expect a definition of the Feynman path integral (Hamiltonian and Lagrangian formulation) and an indication of its origin; a discussion of how it connects to quantum propagator; connection to classical statistical mechanics; and to the quantum partition function; E. g. Harmonic oscillator; strings.

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6. The first part of this question is again book-work. Expect a statement of the Feynman path integral for the double well. An account of the instanton method should explain the utility of the Euclidean time representation, and the inverted potential; a qualitative description of the bounce saddle-point together with an indication of the relative time scales (i.e. what sets the width of the bounce — ω_0^{-1} required in the second part of the problem); a description of the free instanton gas; and finally, an indication how this formulation leads to the quantum splitting of the degenerate minima. Extra credit will be given for a description of the zero mode.

   The second part of the question is indeed quite straightforward although it may look challenging. By parametrising the instanton/anti-instanton pair as a top-hat with smoothed edges (taken from first part of the problem), it is a trivial matter to show that the integral generates the logarithmic potential. Students who would encounter difficultly in understanding what approximations to employ should again be able to draw on the answer to figure out what is happening.

   The final part of the problem involves the realisation that this term in the action can lead to a confinement of the instanton pairs and a complete suppression of tunnelling — zero temperature quantum phase transition. I.e. typical separation is given by ($S_{\rm part}$ is constant, independent of τ)

$$\langle\tau\rangle \sim \int d\tau\tau\,(\omega_0\tau)^{-\eta q_0^2/\pi} \to \begin{cases}\infty & \eta q_0^2/2\pi < 1,\\ \text{const.} & \eta q_0^2/2\pi > 1.\end{cases}$$

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7. The phenomenology of Ginzburg-Landau theory is based on the divergence of the correlation length in the vicinity of a second-order phase transition. This implies that singular critical properties of the theory depend only on fundamental symmetry properties of the model and not on the microscopic details of the Hamiltonian. This include locality, translational or rotational invariance, and scale invariance.

(a) In the mean-field approximation, the average magnetisation takes the homogeneous form $\mathbf{m} = \bar{m}\hat{\mathbf{e}}_\ell$ where $\hat{\mathbf{e}}_\ell$ represents a unit vector along some arbitrary direction, and $\bar{m}$ minimises the free energy density

$$f(\bar{m}) = \frac{\beta H[\bar{m}]}{V} = \frac{t}{2}\bar{m}^2 + u\bar{m}^4.$$

Differentiating, we find that

$$\bar{m} = \begin{cases}0 & t > 0,\\ \sqrt{-t/4u} & t < 0.\end{cases}$$

(b) Applying the expansion, and making use of the identities

$$\begin{aligned}(\nabla\mathbf{m})^2 &= (\nabla\phi_\ell)^2 + (\nabla\phi_t^i)^2\\ \mathbf{m}^2 &= \bar{m}^2 + 2\bar{m}\phi_\ell + \phi_\ell^2 + (\phi_t^i)^2\\ \left(\mathbf{m}^2\right)^2 &= \bar{m}^4 + 4\bar{m}^3\phi_\ell + 6\bar{m}^2\phi_\ell^2 + 2\bar{m}^2(\phi_t^i)^2 + O(\phi^3)\end{aligned}$$

we find

$$\beta H = \beta H[\bar{m}] + \int d^2\mathbf{x}\frac{K}{2}\left[(\nabla\phi_\ell)^2 + \xi_\ell^{-2}\phi_\ell^2 + (\nabla\phi_t^i)^2 + \xi_t^{-2}(\phi_t^i)^2\right]$$

where

$$\frac{K}{\xi_\ell^2} = \begin{cases}t & t > 0\\ -2t & t < 0\end{cases}, \qquad \frac{K}{\xi_t^2} = \begin{cases}t & t > 0\\ 0 & t < 0\end{cases}$$

(c) Expressed in Fourier representation, the Hamiltonian is diagonal and the transverse correlation function is given by

$$\langle\phi_t^i(\mathbf{q})\phi_t^j(\mathbf{q}')\rangle = (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\delta_{ij}\frac{1}{K(\mathbf{q}^2 + \xi_t^{-2})}$$

Turning to the real space correlation function, for $t < 0$

$$\langle\phi_t^i(\mathbf{x})\phi_t^j(0)\rangle = \int\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{K\mathbf{q}^2} \sim \frac{1}{K|\mathbf{x}|^{d-2}}.$$

In dimensions $d > 2$ the correlation function decays at large distances while in dimensions $d \leq 2$ the correlation function diverges. This is consistent with the Mermin-Wagner theorem which implies the destruction of long-range order due to Goldstone mode fluctuations.

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8. Essay question: Refer to lecture notes.

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9. Switching to the momentum basis

$$\theta(\mathbf{q}) = \int d^2\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}}\theta(\mathbf{r}), \qquad \theta(\mathbf{r}) = \int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{r}}\theta(\mathbf{q}),$$

the Hamiltonian takes the form

$$\beta H = \frac{J}{2}\int \frac{d^2\mathbf{q}}{(2\pi)^2}\mathbf{q}^2|\theta(\mathbf{q})|^2$$

According to this result there exist low energy massless fluctuations of the field θ known as a Goldstone modes. The influence of these fluctuations on the long-range order in the system can be estimated by calculating the autocorrelator.

In the momentum basis, the autocorrelator of phases takes the form

$$\langle\theta(\mathbf{q}_1)\theta(\mathbf{q}_2)\rangle = (2\pi)^2\delta^2(\mathbf{q}_1+\mathbf{q}_2)\frac{1}{J\mathbf{q}_1^2}.$$

from which we obtain the real space correlator

$$\begin{aligned}\left\langle(\theta(\mathbf{r})-\theta(\mathbf{0}))^2\right\rangle &= \int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{|1-e^{i\mathbf{q}\cdot\mathbf{r}}|^2}{J\mathbf{q}^2} = 4\int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\sin^2(\mathbf{q}\cdot\mathbf{r})}{J\mathbf{q}^2}\\ &= \frac{1}{\pi J}\ln\left(\frac{|\mathbf{r}|}{a}\right)\end{aligned}$$

where a represents some lower length scale cut-off. From this result, we see that the correlation function decays as a power law in two-dimensions corresponding to quasi-long range order. This is in accord with the Mermin-Wagner theorem which states that the breaking of a spontaneous symmetry is accompanied by the existence of massless Goldstone modes which destroy long-range order in dimensions $d \leq 2$.

(a) A vortex configuration of unit charge is defined by

$$\partial\theta(\mathbf{r}) = \frac{1}{|\mathbf{r}|}\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z$$

Substituting this expression into the effective free energy, we obtain the vortex energy

$$\beta E_{\text{vortex}} = \frac{J}{2}\int\frac{d^2\mathbf{r}}{\mathbf{r}^2} = \pi J\ln\left(\frac{L}{a}\right) + \beta E_{\text{core}}$$

where a represents some short-distance cut-off and $\beta E_{\rm core}$ denotes the core energy.

(b) According to the harmonic fluctuations of the phase field, long-range order is destroyed at any finite temperature. However, the power law decay of correlations is consistent with the existence of quasi-long range order. The condensation of vortices indicates a phase transition to a fully disordered phase. An estimate for this melting temperature can be obtained from the single vortex configuration. Taking into account the contribution of a single vortex configuration to the partition function we have

$$\mathcal{Z} \sim \left(\frac{L}{a}\right)^2 e^{-\beta E_{\rm vortex}}$$

where the prefactor is an estimate of the entropy. The latter indicates a condensation of vortices at a temperature $J = 2/\pi$.

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10. In a second order phase transition an order parameter grows continuously from zero. The onset of order below the transition is accompanied by a spontaneous symmetry breaking — the symmetry of the low temperature ordered phase is lower than the symmetry of the high temperature disordered phase. An example is provided by the classical ferromagnet where the appearance of net magnetisation breaks the symmetry $m \mapsto -m$. If the symmetry is continuous, the spontaneous breaking of symmetry is accompanied by the appearance of massless Goldstone mode excitations. In the magnet, these excitations are known as spin waves.

   (a) Applying the rules of Gaussian functional integration, one finds that $\langle\theta(\mathbf{x})\rangle = 0$, and the correlation function takes the form

   $$G(\mathbf{x},\mathbf{x}') \equiv \langle\theta(\mathbf{x})\theta(0)\rangle = -\frac{C_d(\mathbf{x})}{\bar{K}}, \qquad \nabla^2 C_d(\mathbf{x}) = \delta^d(\mathbf{x})$$

   where C_d denotes the Coulomb potential for a δ-function charge distribution. Exploiting the symmetry of the field, and employing Gauss', $\int d\mathbf{x}\, \nabla^2 C_d(\mathbf{x}) = \oint dS \cdot \nabla C_d$, one finds that C_d depends only on the radial coordinate x, and

   $$\frac{dC_d}{dx} = \frac{1}{x^{d-1}S_d}, \qquad C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + \text{const.},$$

   where $S_d = 2\pi^{d/2}/(d/2-1)!$ denotes the total d-dimensional solid angle.

   (b) Using this result, one finds that

   $$\left\langle[\theta(\mathbf{x})-\theta(0)]^2\right\rangle = 2\left[\langle\theta(0)^2\rangle - \langle\theta(\mathbf{x})\theta(0)\rangle\right] \stackrel{|\mathbf{x}|>a}{=} \frac{2(|\mathbf{x}|^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d},$$

   where the cut-off, a is of the order of the lattice spacing. (Note that the case where $d = 2$, the combination $|\mathbf{x}|^{2-d}/(2-d)$ must be interpreted as $\ln|\mathbf{x}|$.

This result shows that the long distance behaviour changes dramatically at $d = 2$. For $d > 2$, the phase fluctuations approach some finite constant as $|\mathbf{x}| \to \infty$, while they become asymptotically large for $d \leq 2$. Since the phase is bounded by 2π, it implies that long-range order (predicted by the mean-field theory) is destroyed.

Turning to the two-point correlation function of $\mathbf{m}$, and making use of the Gaussian functional integral, obtains

$$\langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0) \rangle = \bar{m}^2 \mathrm{Re} \, \left\langle e^{i[\theta(\mathbf{x})-\theta(0)]} \right\rangle .$$

For Gaussian distributed variables $\langle \exp[\alpha\theta] \rangle = \exp[\alpha^2 \langle \theta^2 \rangle / 2]$.

We thus obtain

$$\langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0) \rangle = \bar{m}^2 \exp\left[-\frac{1}{2} \langle [\theta(\mathbf{x}) - \theta(0)]^2 \rangle \right] = \bar{m}^2 \exp\left[-\frac{(|\mathbf{x}|^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d} \right],$$

implying a power-law decay of correlations in $d = 2$, and an exponential decay in $d < 2$. From this result we find

$$\lim_{|\mathbf{x}|\to\infty} \langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0) \rangle = \begin{cases} m_0^2 & d > 2, \\ 0 & d \leq 2. \end{cases}$$

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11. (a) A second order phase transition is associated with the continuous development of an order parameter. At the critical point, various correlation functions are seen to exhibit singular behaviour. In particular, the correlation length, the typical scale over which fluctuations are correlated, diverges. This fact motivates the consideration of a coarse-grained phenomenology in which the microscopic details of the system are surrendered.

To this end, one introduces a coarse-grained order parameter which varies on a length scale much greater than the microscopic scales. The effective Hamiltonian, is phenomenological depending only on the fundamental symmetries of the system.

A general d-dimensional theory involves an n-component order parameter $\mathbf{m}(\mathbf{x})$, i.e.

$$\mathbf{x} \equiv (x_1, \cdots x_d) \in R^d \quad \text{(space)}, \qquad \mathbf{m} \equiv (m_1, \cdots m_n) \in R^n \quad \text{(orderparameter)}.$$

Some specific problems covered in this framework include:

- $n = 1$: Liquid-gas transitions; binary mixtures; and uniaxial magnets;
- $n = 2$: Superfluidity; superconductivity; and planar magnets;
- $n = 3$: Classical isotropic magnets.

While most applications occur in three-dimensions, there are also important phenomena on surfaces ($d = 2$), and in wires ($d = 1$). (Relativistic field theory is described by a similar structure, but in $d = 4$.)

A general coarse-grained effective Hamiltonian can be constructed on the basis of appropriate symmetries:

(a) **Locality**: The Hamiltonian should depend on the local magnetisation and short range interactions expressed through gradient expansions:

$$\beta H = \int d\mathbf{x}\ f[\mathbf{m}(\mathbf{x}), \nabla\mathbf{m}, \cdots]$$

(b) **Rotational Symmetry**: Without magnetic field, the Hamiltonian should be isotropic in space and therefore invariant under rotations, $\mathbf{m} \mapsto \mathbf{R}_n\mathbf{m}$.

$$\beta H[\mathbf{m}] = \beta H[\mathbf{R}_n\mathbf{m}].$$

(c) **Translational and Rotational Symmetry in x**: This last constraint finally leads to a Hamiltonian of the form

$$\beta H = \int d\mathbf{x}\ \Big[\frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + \cdots \\ + \frac{K}{2}(\nabla\mathbf{m})^2 + \frac{L}{2}(\nabla^2\mathbf{m})^2 + \frac{N}{2}\mathbf{m}^2(\nabla\mathbf{m})^2 + \cdots - \mathbf{h}\cdot\mathbf{m}\Big]$$

This final result is known as the Ginzburg-Landau Hamiltonian. It depends on a set of phenomenological parameters t, u, K, etc. which are non-universal functions of microscopic interactions, as well as external parameters such as temperature, and pressure.

With this definition, the total partition function for the system is given by the functional integral over the field configurations of the order parameter weighted by the corresponding Boltzmann weight

$$\mathcal{Z} = \int D\mathbf{m}(\mathbf{x}) e^{-\beta H[\mathbf{m}(\mathbf{x})]}.$$

(b) According to the Mermin-Wagner theorem, spontaneous symmetry breaking of a continuous symmetry leads to the appearance of Goldstone modes which destroy long-range order in dimensions $d \leq 2$. However, in two-dimensions, there exists a low temperature phase of quasi long-range order in which the correlations decay algebraically at long-distances. This leaves open the room for a phase transition at some intermediate temperature in which the correlation function crosses over to exponential decay.

To understand the nature of the transition, it is necessary to take into account the existence of topological defects, vortex configurations of the fields. The elementary defect which has a unit charge involves a 2π twist of θ as one encircles the defect. More formally,

$$\oint \nabla\theta \cdot d\ell = 2\pi n \quad \Longrightarrow \quad \nabla\theta = \frac{n}{r}\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z,$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_z$ are unit vectors respectively in the plane and perpendicular to it. This (continuum) approximation fails close to the centre (core) of the vortex, where the lattice structure is important.

The energy cost from a single vortex of charge n has contributions from the core region, as well as from the relatively uniform distortions away from the centre. The distinction between regions inside and outside the core is arbitrary, and for simplicity, we shall use a circle of radius a to distinguish the two, i.e.

$$\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a d^2\mathbf{x}(\nabla\theta)^2 = \beta E_n^0(a) + \pi K n^2 \ln\left(\frac{L}{a}\right).$$

The dominant part of the energy comes from the region outside the core and diverges logarithmically with the system size L. The large energy cost associated with the defects prevents their spontaneous formation close to zero temperature. The partition function for a configuration with a single vortex of charge n is

$$\mathcal{Z}_1(n) \approx \left(\frac{L}{a}\right)^2 \exp\left[-\beta E_n^0(a) - \pi K n^2 \ln\left(\frac{L}{a}\right)\right],$$

where the factor of $(L/a)^2$ results from the configurational entropy of possible vortex locations in an area of size L^2. The entropy and energy of a vortex both grow as $\ln L$, and the free energy is dominated by one or the other. At low temperatures, large K, energy dominates and $\mathcal{Z}_1$, a measure of the weight of configurations with a single vortex, vanishes. At high enough temperatures, $K < K_n = 2/(\pi n^2)$, the entropy contribution is large enough to favour spontaneous formation of vortices. On increasing temperature, the first vortices that appear correspond to $n = \pm 1$ at $K_c = 2/\pi$. Beyond this point many vortices appear and the equation above is no longer applicable.

In fact this estimate of K_c represents only a lower bound for the stability of the system towards the condensation of topological defects. This is because pairs (dipoles) of defects may appear at larger couplings. Consider a pair of charges ± 1 separated by a distance d. Distortions far from the core $|\mathbf{r}| \gg d$ can be obtained by superposing those of the individual vortices

$$\nabla\theta = \nabla\theta_+ + \nabla\theta_- \approx 2\mathbf{d}\cdot\nabla\left(\frac{\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z}{|\mathbf{r}|}\right),$$

which decays as $d/|\mathbf{r}|^2$. Integrating this distortion leads to a finite energy, and hence dipoles appear with the appropriate Boltzmann weight at any temperature. The low temperature phase should therefore be visualised as a gas of tightly bound dipoles, their density and size increasing with temperature. The high temperature phase constitutes a plasma of unbound vortices.

(c) The divergence of the correlation length at a second order phase transition suggests that in the vicinity of the transition, the microscopic lengths are irrelevant. The critical behaviour is dominated by fluctuations that are statistically self-similar up to the scale ξ. Self-similarity allows the gradual elimination of the correlated degrees of freedom at length scales $x \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at scale ξ. This is achieved through a procedure known as the Renormalisation Group (RG), whose conceptual foundation is outlined below:

(a) *Coarse-Grain*: The first step of the RG is to decrease the resolution by changing the minimum length scale from the microscopic scale a to ba where $b > 1$. The coarse-grained magnetisation is then

$$\bar{\mathbf{m}}(\mathbf{x}) = \frac{1}{b^d}\int_{\text{Cell centred at }\mathbf{x}} d\mathbf{y}\ \mathbf{m}(\mathbf{y}).$$

(b) *Rescale*: Due to the change in resolution, the coarse-grained "picture" is grainier than the original. The original resolution a can be restored by decreasing all length scales by a factor b, i.e. defining

$$\mathbf{x}' = \frac{\mathbf{x}}{b}.$$

Thus, at each position $\mathbf{x}'$ we have defined an average moment $\bar{\mathbf{m}}(\mathbf{x}')$.

(c) *Renormalise*: The relative size of the fluctuations of the rescaled magnetisation profile is in general different from the original, i.e. there is a change in contrast between the pictures. This can be remedied by introducing a factor ζ, and defining a renormalised magnetisation

$$\mathbf{m}'(\mathbf{x}') = \frac{1}{\zeta}\bar{\mathbf{m}}(\mathbf{x}').$$

The choice of ζ will be discussed later.

By following these steps, for each configuration $\mathbf{m}(\mathbf{x})$ we generate a renormalised configuration $\mathbf{m}'(\mathbf{x}')$. It can be regarded as a mapping of one set of random variables to another, and can be used to construct the probability distribution. Kadanoff's insight was to realise that since, on length scales less than ξ, the renormalised configurations are statistically similar to the original ones, they must be distributed by a Hamiltonian that is also close to the original. In particular, if the original Hamiltonian βH is at a critical point, $t = h = 0$, the new $(\beta H)'$ is also at criticality since no new length scale is generated in the renormalisation procedure, i.e. $t' = h' = 0$.

However, if the Hamiltonian is originally off criticality, then the renormalisation takes us further away from criticality because $\xi' = \xi/b$ is smaller. The next assumption is that since any transformation only involves changes at the shortest length scales it can not produce singularities. The renormalised parameters must be *analytic* functions, and hence expandable as

$$\begin{cases} t(b;t,h) = A(b)t + B(b)h + O(t^2,h^2,th), \\ h(b;t,h) = C(b)t + D(b)h + O(t^2,h^2,th). \end{cases}$$

However, the known behaviour at $t = h = 0$ rules out a constant term in the expansion, and to prevent a spontaneously broken symmetry we further require $C(b) = B(b) = 0$. Finally, commutativity $A(b_1 \times b_2) = A(b_1) \times A(b_2)$ implies $A(b) = b^{y_t}$ and $D(b) = b^{y_h}$. So, to lowest order

$$\begin{cases} t_b \equiv t(b) = b^{y_t} t, \\ h_b \equiv h(b) = b^{y_h} h. \end{cases}$$

where y_t, $y_h > 0$. As a consequence, since the statistical weight of new configuration, $W'[\mathbf{m}']$ is the sum of the weights $W[\mathbf{m}]$ of old ones, the partition function is preserved

$$\mathcal{Z} = \int D\mathbf{m}\; W[\mathbf{m}] = \int D\mathbf{m}'\; W'[\mathbf{m}'] = \mathcal{Z}'.$$

From this is follows that the free energies density takes the form

$$f(t,h) = -\frac{\ln \mathcal{Z}}{V} = -\frac{\ln \mathcal{Z}'}{V' b^d} = b^{-d} f(t_b, h_b) = b^{-d} f(b^{y_t} t, b^{y_h} h),$$

where we have assumed that the two free energies are obtained from the *same Hamiltonian* in which only the parameters t and h have changed. The free energy describes a homogeneous function of t and h. This is made apparent by choosing a rescaling factor b such that $b^{y_t} t$ is a constant, say unity, i.e. $b = t^{-1/y_t}$, and

$$f(t,h) = t^{d/y_t} f(1, h/t^{y_h/y_t}) \equiv t^{d/y_t} g_f(h/t^{y_h/y_t})$$

(d) Starting from the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle$$

the time evolution operator is defined by

$$|\Psi(t')\rangle = \hat{U}(t',t)|\Psi(t)\rangle, \qquad \hat{U}(t',t) = \exp\left[-\frac{i}{\hbar}\hat{H}(t'-t)\right].$$

In the real space representation

$$U(x',t';x,t) = \langle x'|\exp\left[-\frac{i}{\hbar}\hat{H}(t'-t)\right]|x\rangle,$$

According to the Feynman path integral, the quantum evolution operator is expressed as the sum over all trajectories subject to the boundary conditions and weighted by the classical action. In the Hamiltonian formulation,

$$U(x',t';x,t) = \int Dx(t) \int Dp(t) \exp\left[\frac{i}{\hbar} S(p,x)\right],$$
$$S(p,x) = \int_t^{t'} dt'' \left[p\dot{x} - H(p,x)\right],$$

and in the Lagrangian formulation,

$$U(x',t';x,t) = \int \bar{D}x(t) \exp\left[\frac{i}{\hbar} S(x)\right],$$
$$S(x) = \int_t^{t'} dt'' \left[\frac{m}{2}\dot{x}^2 - V(x)\right].$$

To establish an analogy with statistical mechanics we have to consider propagation in *imaginary* or *Euclidean time* T. In this way, we obtain

$$U(x', t' = -iT; x, t = 0) = \int Dx(\tau) \exp\left[-\frac{1}{\hbar} S(x)\right],$$

where

$$S(x) = \int_0^T d\tau \left[\frac{m}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + V(x(\tau))\right].$$

Interpreting the action as a classical free energy functional, and the path integral as a classical partition function, one has the analogy: The transition amplitude for a quantum particle for the time (-iT) is equal to the classical partition function for a string of length T computed at the value $\beta = 1/\hbar$.

A second analogy follows from the fact that the *quantum* partition function for the particle is given by $\mathcal{Z}_{\rm qu} = \mathrm{Tr}\exp[-\beta H]$ and hence,

$$\mathcal{Z}_{\rm qu} = \int dx \langle x|e^{-\beta H}|x\rangle = \int dx U(x, t' = -i\beta\hbar; x, t = 0).$$

Therefore, in quantum statistical mechanics, the inverse temperature plays the role of an imaginary time.

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12. In mean-field, the free energy can be shown to take a homogeneous form around a second order transition. According to the scaling hypothesis, when one goes beyond mean-field, homogeneity of the singular form of the free energy (and of any other thermodynamic quantity) retains the homogeneous form

$$f_{\rm sing.}(t, h) = t^{2-\alpha} g_f(h/t^\Delta)$$

where the actual exponents α and Δ depend on the critical point being considered. (Additional credit for discussion of hyperscaling.)

(a) From the free energy, one obtains the magnetisation as

$$m(t, h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^\Delta).$$

In the limit $x \to 0$, $g_m(x)$ is a constant, and $m(t, h = 0) \sim t^{2-\alpha-\Delta}$ (i.e. $\beta = 2 - \alpha - \Delta$). On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and $m(t = 0, h) \sim t^{2-\alpha-\Delta}(h/t^\Delta)^p$. Since this limit is independent of t, we must have $p\Delta = 2 - \alpha - \Delta$. Hence $m(t = 0, h) \sim h^{(2-\alpha-\Delta)/\Delta}$ (i.e. $\delta = \Delta/(2 - \alpha - \Delta) = \Delta/\beta$).

(b) From the magnetisation, one obtains the susceptibility

$$\chi(t, h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^\Delta) \Rightarrow \chi(t, h = 0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha.$$

(c) Close to criticality, the correlation length ξ is solely responsible for singular contributions to thermodynamic quantities. Since $\ln \mathcal{Z}(t,h)$ is dimensionless and extensive (i.e. $\propto L^d$), it must take the form

$$\ln \mathcal{Z} = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). (A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$. The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t,h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^{\Delta})$$

As a consequence, comparing with the homogeneous expression for the free energy, one obtains the Josephson identity

$$2 - \alpha = d\nu$$

---

13. (a,b,c) Expanding the Hamiltonian to cubic order in the longitudinal and transverse fluctuations, we obtain

$$\begin{aligned}\beta H[\phi] - \beta H[0] &= \int d^d\mathbf{x} \Big\{ \frac{K}{2}\left[(\nabla\phi_l)^2 + (\nabla\phi_t)^2\right] + \left[\frac{t}{2} + 4u\bar{m}^2\right] \\ &\quad + \left[\frac{t}{2} + 2u\bar{m}^2\right]\phi_t^2 + 4u\bar{m}\left(\phi_l^3 + \phi_l\phi_t^2\right) + \cdots \Big\},\end{aligned}$$

where, for $t<0$, $\bar{m}^2 = |t|/4u$, and for $t>0$, $\bar{m}=0$. Thus, for $t<0$, we obtain

$$\beta H[\phi] - \beta H[0] = \int d^d\mathbf{x} \Big\{ \frac{K}{2}\left[(\nabla\phi_l)^2 + (\nabla\phi_t)^2\right] + \frac{|t|}{2}\phi_l^2 + 4u\bar{m}\phi_l\phi_t^2 + \cdots \Big\}.$$

Keeping only the quadratic order, the bare correlators takes the form

$$\begin{aligned}\langle \phi_l(\mathbf{q}_1)\phi_l(\mathbf{q}_2)\rangle_0 &= (2\pi)^d \delta^d(\mathbf{q}_1 + \mathbf{q}_2)\frac{1}{K\mathbf{q}_1^2 + |t|} \\ \left\langle \phi_t^\alpha(\mathbf{q}_1)\phi_t^\beta(\mathbf{q}_2)\right\rangle_0 &= (2\pi)^d \delta^d(\mathbf{q}_1 + \mathbf{q}_2)\frac{1}{K\mathbf{q}_1^2}.\end{aligned}$$

From this result, we obtain the momentum dependent susceptibility

$$\chi_l = \frac{1}{K\mathbf{q}^2 + |t|}, \qquad \chi_t = \frac{1}{K\mathbf{q}^2}.$$

Treating the cubic interaction as a perturbation, we obtain the cumulant expansion

$$\begin{aligned}\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle &= \frac{\left\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')e^{-U}\right\rangle_0}{\left\langle e^{-U}\right\rangle_0}\\ &= \langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle_0 + \frac{1}{2}\left[\left\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')U^2\right\rangle_0 - \langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle\left\langle U^2\right\rangle_0\right] + \cdots\end{aligned}$$

(d) Making use of the correlator

$$\begin{aligned}\langle\phi_t(\mathbf{q}_1)\cdot\phi_t(\mathbf{q}_2)\phi_t(\mathbf{q}_1')\cdot\phi_t(\mathbf{q}_2')\rangle_0 &= \frac{(2\pi)^{2d}}{K^2}\Big[(n-1)^2\frac{\delta^d(\mathbf{q}_1+\mathbf{q}_2)\delta^d(\mathbf{q}_1'+\mathbf{q}_2')}{\mathbf{q}_1^2\mathbf{q}_1'^2}\\ +(n-1)\frac{\delta^d(\mathbf{q}_1+\mathbf{q}_1')\delta^d(\mathbf{q}_2+\mathbf{q}_2')}{\mathbf{q}_1^2\mathbf{q}_2^2} &+ (n-1)\frac{\delta^d(\mathbf{q}_1+\mathbf{q}_2')\delta^2(\mathbf{q}_2+\mathbf{q}_1')}{\mathbf{q}_1^2\mathbf{q}_2^2}\Big],\end{aligned}$$

we obtain

$$\begin{aligned}\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle - \langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle_0 = \frac{16(u\bar{m})^2}{2}\int(d\mathbf{q}_1)(d\mathbf{q}_2)(d\mathbf{q}_3)(d\mathbf{q}_4)\\ \times(\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\phi_l(-\mathbf{q}_1-\mathbf{q}_2)\phi_l(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0\\ -\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle_0\langle\phi_l(-\mathbf{q}_1-\mathbf{q}_2)\phi_l(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0)\\ \times\langle\phi_t(\mathbf{q}_1)\cdot\phi_t(\mathbf{q}_2)\phi_t(\mathbf{q}_1')\cdot\phi_t(\mathbf{q}_2')\rangle_0\end{aligned}$$

Using the cumulant

$$\begin{aligned}&\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\phi_l(-\mathbf{q}_1-\mathbf{q}_2)\phi_l(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0\\ &-\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle_0\langle\phi_l(-\mathbf{q}_1-\mathbf{q}_2)\phi_l(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0\\ &=\langle\phi_l(\mathbf{q})\phi_l(-\mathbf{q}_1-\mathbf{q}_2)\rangle_0\langle\phi_l(\mathbf{q}')\phi_l(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0\\ &\quad+\langle\phi_l(\mathbf{q})\phi_l(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0\langle\phi_l(\mathbf{q}')\phi_l(-\mathbf{q}')\phi_l(-\mathbf{q}_1-\mathbf{q}_2)\rangle_0,\end{aligned}$$

we obtain

$$\begin{aligned}\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle - \langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle_0 = \frac{2u|t|(2\pi)^d}{K^2(K\mathbf{q}^2+|t|)(K\mathbf{q}'^2+|t|)}\\ \times\left[(n-1)^2\delta^d(\mathbf{q})\delta^d(\mathbf{q}')\int\frac{(d\mathbf{q}_1)(d\mathbf{q}_2)}{\mathbf{q}_1^2\mathbf{q}_1'^2} + 2(n-1)\delta^d(\mathbf{q}+\mathbf{q}')\int\frac{(d\mathbf{q}_1)}{(\mathbf{q}+\mathbf{q}_1)^2\mathbf{q}_1^2}\right].\end{aligned}$$

Note that the last term is infrared divergent in dimensions $d<4$.

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14. Essay question: Refer to lecture notes.

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15. (a) Applying the Hubbard-Stratonovich transformation,

$$\exp\left[\frac{1}{2}\sum_{ij}\sigma_i J_{ij}\sigma_j\right] = C\int_{-\infty}^{\infty}\prod_k dm_k\exp\left[-\frac{1}{2}\sum_{ij}m_i[J^{-1}]_{ij}m_j + \sum_i m_i\phi_i\right],$$

and summing over the spin configurations, we obtain

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\frac{1}{2}\sum_{ij} m_i[J^{-1}]_{ij}m_j + \sum_i \ln\left(2\cosh(2(m_i+h))\right)\right].$$

To determine J^{-1} it is convenient to switch to the basis in which J is diagonal — reciprocal space. Defining the Fourier series

$$\sigma(\mathbf{q}) = \sum_n e^{i\mathbf{q}\cdot\mathbf{n}}\sigma_{\mathbf{n}}, \qquad \sigma_{\mathbf{n}} = \int_{-\pi}^{\pi}(d\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{n}}\sigma(\mathbf{q}),$$

we obtain

$$\frac{1}{2}\sum_{ij}\sigma_i J_{ij}\sigma_j = \int (d\mathbf{q})J(\mathbf{q})|\sigma(\mathbf{q})|^2, \qquad J(\mathbf{q}) = \frac{J}{2}\sum_{\hat{e}} e^{i\mathbf{q}\cdot\hat{e}}$$

where $\hat{e}$ denote the primitive lattice vectors. From this result, we obtain the expansion

$$J(\mathbf{q}) = J\sum_{d=1}^{D}\cos q_d \approx D - \frac{\mathbf{q}^2}{2} + \cdots$$

Inverting and applying the inverse Fourier transform, we obtain

$$\begin{aligned}[J^{-1}]_{ij} &= \int (d\mathbf{q})\frac{e^{i\mathbf{q}\cdot(\mathbf{n}_i-\mathbf{n}_j)}}{J(\mathbf{q})} \approx \frac{1}{J}\int (d\mathbf{q})e^{i\mathbf{q}\cdot(\mathbf{n}_i-\mathbf{n}_j)}\left(D^{-1} + \frac{\mathbf{q}^2}{2D} + \cdots\right) \\ &= \frac{1}{DJ}\delta_{\mathbf{n}_i,\mathbf{n}_j} - \frac{1}{2D^2J}\nabla^2_{\mathbf{n}_i-\mathbf{n}_j} + \cdots\end{aligned}$$

Applying this expansion, we obtain

$$\mathcal{Z} \approx C\int Dm \exp\left[-\frac{1}{2}\int d^d\mathbf{x}\left(\frac{m^2}{DJ} + \frac{1}{2D^2J}(\nabla m)^2\right) + \int d^d\mathbf{x}\ln\left(2\cosh(2(m+h))\right)\right].$$

Expanding the logarithm for small h and m we obtain

$$\mathcal{Z} = \int Dm e^{-\beta H},$$

where the effective Ginzburg-Landau Hamiltonian takes the form

$$\beta H = \int d^d\mathbf{x}\left(\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 - hm\right)$$

with

$$K = \frac{1}{4D^2J}, \qquad t = \frac{1}{2DJ} - 1, \qquad u = \frac{1}{12}.$$

Setting $t = 0$, we establish the critical point at $J_c^{-1} \equiv T_c^{-1} < 2D$.

(b-d) An estimate of the mean-field properties of the Ginzburg-Landau Hamiltonian is straightforward and can be found in the lecture notes.

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16. (a) Expressed as a Euclidean time path integral, the transition probability is given by

$$\mathcal{Z} = \int Dq(\tau) e^{-S[q(\tau)]/\hbar}, \qquad S[q] = \int_0^\infty d\tau \left[\frac{m}{2}\dot{q}^2 + U(q)\right],$$

where the confining potential takes the form

$$U(q) = 2g\sin^2\left(\frac{\pi q}{q_0}\right).$$

The corresponding classical equation of motion is given by the *Sine-Gordon Equation*

$$m\ddot{q} - \frac{2\pi g}{q_0}\sin\left(\frac{2\pi q}{q_0}\right) = 0.$$

Applying the trial solution

$$\bar{q}(\tau) = \frac{2q_0}{\pi}\tan^{-1}\left(e^{\omega_0 \tau}\right),$$

we find the equation is satisfied if $\omega_0 = (2\pi/q_0)\sqrt{g/m}$.

From this result we obtain the "classical" action

$$\begin{aligned} S[\bar{q}] &= \int_0^\infty d\tau \left[\frac{m}{2}\dot{\bar{q}}^2 + U(\bar{q})\right] = \int_0^\infty d\tau m\dot{\bar{q}}^2 = m\int_0^{q_0} dq\dot{\bar{q}} \\ &= 2\frac{mq_0^2}{\pi^2}\omega_0. \end{aligned}$$

(b) In Fourier space, the action of the classical string takes the form

$$S_{\text{string}} = \int_{-\infty}^\infty \frac{d\omega}{2\pi}\int_{-\infty}^\infty \frac{dk}{2\pi}\frac{1}{2}(\rho\omega^2 + \sigma k^2)|u(\omega,k)|^2.$$

Representing the functional δ-function as the integral

$$\int Df \exp\left[i\int_{-\infty}^\infty \frac{d\omega}{2\pi} f(\omega)\left(q(-\omega) - \int_{-\infty}^\infty \frac{dk}{2\pi}u(-\omega,-k)\right)\right],$$

and performing the integral over the degrees of freedom of the string, we obtain

$$\int Du e^{-S_{\text{string}}/\hbar - i\int d\tau f(\tau)u(\tau,0)} = \text{const.}\times\exp\left[-\int_{-\infty}^\infty \frac{d\omega}{2\pi}\int_{-\infty}^\infty \frac{dk}{2\pi}\frac{1}{2}\frac{\hbar}{(\rho\omega^2+\sigma k^2)}|f(\omega)|^2\right].$$

Applying the integral

$$\int_{-\infty}^\infty \frac{dk}{2\pi}\frac{1}{\rho\omega^2 + \sigma k^2} = \frac{1}{|\omega|\eta},$$

where $\eta = \sqrt{\rho\sigma}$, and performing the Gaussian functional integral over the Lagrange multiplier $f(\omega)$, we obtain the effective action

$$S_{\rm eff} = S_{\rm part.} + \frac{\eta}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}|\omega||q(\omega)|^2.$$

(c) Approximating the instanton/anti-instanton pair $q(\tau) = \bar{q}(\tau+\bar{\tau}) - \bar{q}(\tau-\bar{\tau})$ by a "top-hat" function we find

$$q(\omega) = \int_{-\bar{\tau}/2}^{\bar{\tau}/2} d\tau q_0 e^{i\omega\tau} = q_0\bar{\tau}\frac{\sin(\omega\bar{\tau}/2)}{(\omega\bar{\tau}/2)}.$$

Treating the dissipative term as a perturbation, we obtain the action

$$\frac{\eta}{2}\int_0^{\omega_0}\frac{d\omega}{2\pi}|\omega|(q_0\bar{\tau})^2\frac{\sin^2(\omega\bar{\tau}/2)}{(\omega\bar{\tau}/2)^2},$$

where ω_0 serves as a high frequency cut-off. Taking $\omega\bar{\tau} \gg 1$ we obtain the approximate integral

$$2\times\frac{\eta}{4}\int_{1/\bar{\tau}}^{\omega_0}\frac{d\omega}{2\pi}q_0^2\frac{4}{\omega} = \frac{q_0^2}{\pi}\eta\ln(\omega_0\bar{\tau}).$$

Employing this result as a probability distribution, we find

$$\langle\bar{\tau}\rangle = \int d\bar{\tau}\bar{\tau}\exp\left[-\frac{q_0^2}{\pi\hbar}\eta\ln(\omega_0\bar{\tau})\right] \sim \text{const.}\times\int^{\infty} d\bar{\tau}\bar{\tau}^{1-q_0^2\eta/\pi\hbar}.$$

The divergence of the integral shows that for

$$\eta > \frac{2\pi\hbar}{q_0^2}$$

instanton/anti-instanton pairs are confined and particle tunneling is prohibited.

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17. (a) In the mean-field approximation (i.e. η is spatially non-varying), by minimising the Free energy density, it is straightforward to show that

$$\bar{\eta} = \begin{cases} 0 & t > 0, \\ (-t/v)^{1/4} & t > 0. \end{cases}, \qquad \beta\bar{F} = \begin{cases} 0 & t > 0, \\ -|t|^{3/2}/3v^{1/2} & t < 0. \end{cases},$$

From this result it is easy to obtain the heat capacity,

$$C_{\rm mf} = -T\frac{\partial^2 f}{\partial t^2} \approx -T_c\frac{\partial^2 f}{\partial t^2} = \begin{cases} 0 & t > 0, \\ (-vt)^{-1/2}T_c/4 & t < 0. \end{cases}$$

(b) Expanding the Hamiltonian to second order in the vicinity of the mean field solution, one finds

$$\beta H(\eta) - \beta H(\bar{\eta}) = \frac{K}{2}\int d^d\mathbf{r}\left[(\nabla\eta)^2 + \frac{\eta^2}{\xi^2}\right], \qquad \frac{1}{\xi^2} = \begin{cases} t/K & t > 0, \\ -4t/K & t < 0. \end{cases}$$

From this result it is straightforward to determine the asymptotic form of the correlation function using the formula given at the end of the question.

$$\langle \eta(0)\eta(\mathbf{r})\rangle = \frac{e^{-|\mathbf{r}|/\xi}}{KS_d|d-2||\mathbf{r}|^{d-2}}$$

This identifies ξ as the correlation length which diverges in the vicinity of the transition.

(c) Again, in the Gaussian approximation, the free energy and heat capacity are easily determined.

$$\beta F = \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left[K(\mathbf{q}^2 + \xi^{-2})\right], \qquad C_{\text{fl}} = C - C_{\text{mf}} \propto K^{-d/2}|t|^{d/2-2}$$

(d) Taking the results for the mean field and fluctuation contribution to the heat capacity, one obtains,

$$\frac{C_{\text{fl}}}{C_{\text{mf}}} \propto \frac{|t|^{(d-3)/2}}{\sqrt{K^d/v}}$$

from which one can identify the upper critical dimension as 3.

(e) Most important difference is the appearance of Goldstone modes due to massless fluctuations of the transverse degrees of freedom. This gives rise to a power law decay of the correlation function below T_c.

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18. (a) In the harmonic approximation

$$\langle h(\mathbf{q}_1)h(\mathbf{q}_2)\rangle = \frac{1}{r_0\mathbf{q}^2 + \kappa_0\mathbf{q}^4}(2\pi)^2\delta^2(\mathbf{q}_1 + \mathbf{q}_2).$$

The corresponding autocorrelation function is given by

$$\langle [h(\mathbf{x}) - h(0)]^2\rangle = \int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{|1 - e^{i\mathbf{q}\cdot\mathbf{x}}|^2}{r_0\mathbf{q}^2 + \kappa_0\mathbf{q}^4} \sim \frac{1}{\pi r_0}\ln\left(q_c|\mathbf{x}|\right),$$

where $q_c \sim \sqrt{r_0/\kappa_0}$ represents the short-distance or ultraviolet cut-off of the integral. The divergence of this function at long distance implies that there is no long-range positional order.

(b) The fluctuation in the normals is given by

$$\langle[\nabla h(\mathbf{x})-\nabla h(0)]^2\rangle=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\mathbf{q}^2|1-e^{i\mathbf{q}\cdot\mathbf{x}}|^2}{r_0\mathbf{q}^2+\kappa_0\mathbf{q}^4}\sim\text{const.}$$

Since this result remains finite (independent of $\mathbf{x}$) in the thermodynamic limit we can deduce that, while there is no long-range positional order, there is long-range orientational order of the membrane in three dimensions.

(c) The general formula for the perturbative RG is just bookwork. Simply writing the identity,

$$\mathcal{Z}=\mathcal{Z}_>\int Dh_<e^{-\beta H_0[h_<]}\left\langle e^{-\beta H_I}\right\rangle_{h_>}$$

and performing a cumulant expansion, one obtains the advertised formula.

(d) Evaluation of the perturbative correction leads to contributions of two non-trivial kinds. The first brings about a renormalisation of the interfacial tension while the second renormalises the bending modulus.

$$\begin{aligned}\langle\beta H_I\rangle_{h_>}\quad\to\quad&\frac{5\kappa_0}{4}\int_0^{\Lambda e^{-\ell}}\frac{d^2\mathbf{q}_1d^2\mathbf{q}_2}{(2\pi)^4}\int_{\Lambda e^{-\ell}}^{\Lambda}\frac{d^2\mathbf{q}_3d^2\mathbf{q}_4}{(2\pi)^4}\\&\times\mathbf{q}_1^2\mathbf{q}_2^2\mathbf{q}_3\cdot\mathbf{q}_4h^<_{\mathbf{q}_1}h^<_{\mathbf{q}_2}\left\langle h^>_{\mathbf{q}_3}h^>_{\mathbf{q}_4}\right\rangle(2\pi)^2\delta^2(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4).\end{aligned}$$

Substituting the form of the propagator, and making use of the approximation

$$\frac{5}{4}\int_{\Lambda e^{-\ell}}^{\Lambda}\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\kappa_0\mathbf{q}^2}{r_0\mathbf{q}^2+\kappa_0\mathbf{q}^4}\approx\frac{5}{4}\frac{1}{2\pi}\Lambda^2(1-e^{-\ell})\frac{\kappa_0\Lambda^2}{r_0\Lambda^2+\kappa_0\Lambda^4}\approx\frac{5\ell}{8\pi},$$

we obtain the renormalisation in the text. Extra credit for providing a diagrammatic representation of the same.

In conclusion, we see that there is a renormalisation of the bending modulus to lower values at longer length scales. The eventual unphysical sign change of the modulus is a signature of the breakdown of the perturbative expansion of the Hamiltonian. It provides an estimate for the length over which the membrane remains approximately rigid (i.e. the persistence length).

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19. Essay question: Refer to lecture notes.

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20. (a) Solving the Schrödinger equation, the wavefunctions obeying periodic boundary conditions take the form $\psi_m=e^{im\theta}$, m integer, and the eigenvalues are given by $E_m=m^2/2I$. Defining the quantum partition function as $\mathcal{Z}=\mathrm{tr}e^{-\beta H}$ we obtain the formula in the text.

(b) Interpreted as a Feynman path integral, the quantum partition function takes the form of a propagator with

$$\mathcal{Z} = \int D\theta(\tau) \exp\left[-\int_0^\beta d\tau \mathcal{L}\right],$$

where the $\mathcal{L} = I\dot{\theta}^2/2$ denotes the Lagrangian. The trace implies that paths $\theta(\tau)$ must start and finish at the same point. However, the translational invariance of the angle in integer multiples of 2π implies the boundary conditions advertised.

(c) Evaluating the partition function, we note

$$\int_0^\beta (\partial_\tau \theta)^2 d\tau \quad = \quad \int_0^\beta \left[\frac{2\pi}{\beta} + \partial_\tau \theta_p\right]^2 d\tau = \beta\left(\frac{2\pi m}{\beta}\right)^2 + \int_0^\beta (\partial_\tau \theta_p)^2 d\tau.$$

Thus, we obtain the partition function

$$\mathcal{Z} = \mathcal{Z}_0 \sum_{m=-\infty}^{\infty} \exp\left[-\frac{I}{2}\frac{(2\pi m)^2}{\beta}\right],$$

where

$$\mathcal{Z}_0 = \int D\theta_p(\tau) \exp\left[-\frac{I}{2}\int_0^\beta (\partial_\tau \theta_p)^2 d\tau\right] = \sqrt{\frac{2\pi I}{\beta}}.$$

denotes the free particle partition function.

(d) Apply the Poisson summation formula with

$$h(x) = \exp\left[-\beta \frac{x^2}{2I}\right],$$

we find

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \exp\left[-\beta\frac{x^2}{2I}\right] &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi \exp\left[-\beta\frac{\phi^2}{2I} + 2\pi i m\phi\right] \\ &= \sqrt{\frac{2\pi I}{\beta}} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{I}{2}\frac{(2\pi m)^2}{\beta}\right].\end{aligned}$$

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21. (a) This question is, to a large extent, bookwork. Part (a) involves direct application of the RG procedure:

<u>Coarse-Grain:</u> The first step of the RG involves the elimination of fluctuations at scales $a < |\mathbf{x}| < ba$ or Fourier modes with wavevectors $\Lambda/b < |\mathbf{q}| < \Lambda$. We

thus separate the fields into slowly and rapidly varying functions, $\mathbf{m}(\mathbf{q}) = \mathbf{m}_>(\mathbf{q}) + \mathbf{m}_<(\mathbf{q})$, where

$$\mathbf{m}(\mathbf{q}) = \begin{cases} \mathbf{m}_<(\mathbf{q}) & 0 < |\mathbf{q}| < \Lambda/b, \\ \mathbf{m}_>(\mathbf{q}) & \Lambda/b < |\mathbf{q}| < \Lambda. \end{cases}$$

Since the Ginzburg-Landau functional is Gaussian, the partition function is separable in the modes and can be reexpressed in the form

$$\mathcal{Z} = \int D\mathbf{m}_<(\mathbf{q}) e^{-\beta H[\mathbf{m}_<]} \int D\mathbf{m}_>(\mathbf{q}) e^{-\beta H[\mathbf{m}_>]}.$$

More precisely, the latter takes the form

$$\mathcal{Z} = \mathcal{Z}_> \int D\mathbf{m}_<(\mathbf{q}) \exp\left[-\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + K\mathbf{q}^2}{2} \right) |\mathbf{m}_<(\mathbf{q})|^2 + \mathbf{h}\cdot\mathbf{m}_<(0) \right],$$

where $\mathcal{Z}_> = \exp[-(nV/2)\int_{\Lambda/b}^{\Lambda} (d^d\mathbf{q}/(2\pi)^d) \ln(t + K\mathbf{q}^2)]$. [Full credit does not require an evaluation of the functional integral over $\mathbf{m}_>$.]

<u>Rescale</u>: The partition function for the modes $\mathbf{m}_<(\mathbf{q})$ is similar to the original, except that the upper cut-off has decreased to Λ/b, reflecting the coarse-graining in resolution. The rescaling, $\mathbf{x}' = \mathbf{x}/b$ in real space, is equivalent to $\mathbf{q}' = b\mathbf{q}$ in momentum space, and restores the cut-off to the original value.

<u>Renormalise</u>: The final step of the RG is the renormalisation of the field, $\mathbf{m}'(\mathbf{x}') = \mathbf{m}_<(\mathbf{x}')/\zeta$. Alternatively, we can renormalise the Fourier modes according to $\mathbf{m}'(\mathbf{q}') = \mathbf{m}_<(\mathbf{q}')/z$, resulting in

$$\begin{aligned} \mathcal{Z} &= \mathcal{Z}_> \int D\mathbf{m}'(\mathbf{q}') e^{-\beta H'[\mathbf{m}'(\mathbf{q}')]}, \\ \beta H' &= \int_0^{\Lambda} \frac{d^d\mathbf{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{t + Kb^{-2}\mathbf{q}'^2}{2} \right) |\mathbf{m}'(\mathbf{q}')|^2 - z\mathbf{h}\cdot\mathbf{m}'(0). \end{aligned}$$

As a result of the RG procedure the set of parameters $\{K, t, h\}$ has transformed from to a new set

$$\begin{cases} K' = Kb^{-d-2}z^2, \\ t' = tb^{-d}z^2, \\ h' = hz. \end{cases}$$

The singular point $t = h = 0$ is mapped onto itself as expected. To make the fluctuations scale invariant at this point, we must ensure that the remaining parameter in the Hamiltonian K stays fixed. This is achieved by the choice $z = b^{1+d/2}$ which implies

$$\begin{cases} t' = b^2 t & y_t = 2, \\ h' = b^{1+d/2} h & y_h = 1 + d/2. \end{cases}$$

For the fixed point $t = t'$, K becomes weaker and the spins become uncorrelated — the high temperature phase.

(b) From these equations, we can predict the scaling of the Free energy

$$\begin{aligned} f_{\text{sing.}}(t,h) &= b^{-d} f_{\text{sing.}}(b^2 t, b^{1+d/2} h), \qquad b^2 t = 1, \\ &= t^{d/2} g_f(h/t^{1/2+d/4}). \end{aligned}$$

[This implies exponents: $2-\alpha = d/2$, $\Delta = y_h/y_t = 1/2 + d/4$, and $\nu = 1/y_t = 1/2$. Comparing with the results from the exact solution we can can confirm the validity of the RG.]

(c) At the fixed point ($t = h = 0$) the Hamiltonian is scale invariant. By dimensional analysis $\mathbf{x} = b\mathbf{x}'$, $\mathbf{m}(\mathbf{x}) = \zeta \mathbf{m}'(\mathbf{x}')$ and

$$(\beta H)^* = \frac{K}{2} b^{d-2} \zeta^2 \int d\mathbf{x}' \, (\nabla \mathbf{m}')^2, \qquad \zeta = b^{1-d/2}.$$

For small perturbations

$$(\beta H)^* + u_p \int d\mathbf{x} \, |\mathbf{m}|^p \to (\beta H)^* + u_p b^d \zeta^p \int d\mathbf{x}' \, |\mathbf{m}'|^p.$$

Thus, in general $u_p \to u'_p = b^d b^{p-pd/2} = b^{y_p} u_p$, where $y_p = p - d(p/2-1)$, in agreement with our earlier findings that $y_1 \equiv y_h = 1 + d/2$ and $y_2 \equiv y_t = 2$. For the Ginzburg-Landau Hamiltonian, the quartic term scales with an exponent $y_4 = 4 - d$ and is therefore relevant for $d < 4$ and irrelevant for $d > 4$. Sixth order perturbations scale with an exponent $y_6 = 6 - 2d$ and is therefore irrelevant for $d > 3$.

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22. (a) Applying a continuum approximation, and expanding the lattice Hamiltonian we obtain

$$\begin{aligned} \beta H &= -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \approx -J \sum_{\langle ij \rangle} \left[1 - \frac{(\theta_i - \theta_j)^2}{2} + \cdots \right] \\ &= -JN + \frac{J}{2} \int d^2\mathbf{r} (\nabla \theta)^2 + \cdots \end{aligned}$$

where we have made use of the approximation $\theta_{i+\hat{e}_x} - \theta_i \sim \nabla \theta_i$.

(b) Recast in the momentum basis

$$\theta(\mathbf{q}) = \int d^2\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} \theta(\mathbf{r}), \qquad \theta(\mathbf{r}) = \int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{r}} \theta(\mathbf{q}),$$

the Hamiltonian takes the form

$$\beta H = -JN + \frac{J}{2} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \mathbf{q}^2 |\theta(\mathbf{q})|^2 + \cdots$$

To quadratic order, the correlator of phases takes the form

$$\langle \theta(\mathbf{q}_1)\theta(\mathbf{q}_2)\rangle = (2\pi)^2\delta^2(\mathbf{q}_1+\mathbf{q}_2)\frac{1}{J\mathbf{q}_1^2}.$$

Using this result, the real space correlator takes the form

$$\begin{aligned}\left\langle (\theta(\mathbf{r})-\theta(\mathbf{0}))^2\right\rangle &= \int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{|1-e^{i\mathbf{q}\cdot\mathbf{r}}|^2}{J\mathbf{q}^2} = 4\int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\sin^2(\mathbf{q}\cdot\mathbf{r})}{J\mathbf{q}^2} \\ &= \frac{1}{\pi J}\ln\left(\frac{|\mathbf{r}|}{a}\right)\end{aligned}$$

where a represents some lower length scale cut-off. Turning to the spin-spin correlation function, we require the average

$$\langle \mathbf{S}(\mathbf{r})\cdot\mathbf{S}(\mathbf{0})\rangle = \left\langle e^{i(\theta(\mathbf{r})-\theta(\mathbf{0}))}\right\rangle = \exp\left[-\frac{1}{2}\left\langle (\theta(\mathbf{r})-\theta(\mathbf{0}))^2\right\rangle\right].$$

Altogether we obtain the result sought in the question. [Note that in this part of the question and the next establishing the numerical prefactor from the integral might not be that easy. However, it is intended that the candidates can deduce these constants from the answer given in the question.]

(c) A vortex configuration of unit charge is defined by

$$\partial\theta(\mathbf{r}) = \frac{1}{|\mathbf{r}|}\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z$$

Substituting this expression into the effective free energy, we obtain the vortex energy

$$\beta E_{\text{vortex}} = \frac{J}{2}\int \frac{d^2\mathbf{r}}{\mathbf{r}^2} = \pi J \ln\left(\frac{L}{a}\right) + \beta E_{\text{core}}$$

where a represents some short-distance cut-off and βE_{core} denotes the core energy.

(d) According to the harmonic fluctuations of the phase field, long-range order is destroyed at any finite temperature. However, the power law decay of correlations is consistent with the existence of quasi-long range order. The condensation of vortices indicates a phase transition to a fully disordered phase. An estimate for this melting temperature can be obtained from the single vortex configuration. Taking into account the contribution of a single vortex configuration to the partition function we have

$$\mathcal{Z} \sim \left(\frac{L}{a}\right)^2 e^{-\beta E_{\text{vortex}}}$$

where the prefactor is an estimate of the entropy. The latter indicates a condensation of vortices at a temperature $J = 2/\pi$.

23. Essay question: Refer to lecture notes.

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24. (a) Applying the Hubbard-Stratonovich transformation, the classical partition function is given by

$$\begin{aligned}
\mathcal{Z} &= \sum_{\{\sigma_i\}} e^{-\beta H[\sigma_i]} = I \sum_{\{\sigma_i\}} \int \prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i [J^{-1}]_{ij} m_j + 2\sum_i \sigma_i m_i + \sum_i h\sigma_i\right] \\
&= I \int \prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i [J^{-1}]_{ij} m_j + \sum_i \ln\left(2\cosh(2m_i + h)\right)\right] \\
&= I \int \prod_{k=1}^{N} dm_k e^{-S},
\end{aligned}$$

where S represents the effective free energy shown in the question.

(b) For small m and h the effective free energy can be expanded as

$$U(m) = -\ln 2 + \frac{t}{2}m^2 + \frac{4}{3}m^4 - 2hm + \cdots$$

where $t = 2(\coth(\kappa/2)/J - 2)$. Evidently, at zero magnetic field, the effective potential $U(m)$ is quartic. For $t < 0$, the potential takes the form of a double well.

The path integral for a particle in a potential well is given by

$$\begin{aligned}
\mathcal{Z} &= \int Dr(t) \exp\left[\frac{i}{\hbar}\int_0^t dt' \left(\frac{m}{2}\dot{r}^2 - U(r)\right)\right] \\
&= \int Dr(\tau) \exp\left[-\frac{1}{\hbar}\int_0^T d\tau' \left(\frac{m}{2}\dot{r}^2 + U(r)\right)\right]
\end{aligned}$$

By identifying r with m, and τ with x, the partition function of the Ising model is seen to be equivalent to the path integral of a particle in a double well potential where the transition time T is equalent to the length of the spin chain L. From this analogy, the magnetic field can be seen as inducing an asymmetry of the potential.

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25. The basis of the Ginzburg-Landau phenomenology rests on the divergence of the correlation length in the vicinity of the critical point. This shows that the nature of the long-distance correlations near the critical point can be described by a phenomenological Hamiltonian which relies only on the fundamental symmetries of the system viz. locality, translational, rotational symmetry, isotropy. Taking into account these constraints one obtains the the Ginzburg-Landau Hamiltonian.

(a) In the mean-field approximation the partition function is dominated by the minimum Hamiltonian. In this approximation one obtains

$$\mathcal{Z} \simeq e^{-\beta F}, \qquad \beta F = \min[\beta H]$$

Symmetry under internal rotations of $\mathbf{m}$ implies that the mean-field involves a spontaneous symmetry breaking. Minimising one obtains the magnitude of the saddle-point mean field to be

$$\bar{m} = \begin{cases} 0 & t > 0 \\ (-t/4u)^{1/2} & t < 0. \end{cases}$$

From this result, we obtain the free energy density

$$f(m) \equiv \frac{\beta F}{V} = \begin{cases} 0 & t > 0 \\ -t^2/16u & t < 0. \end{cases}$$

Then making use of the identity

$$E = -\frac{\partial \ln \mathcal{Z}}{\partial \beta},$$

one obtains the specific heat capacity

$$C_{\text{sing.}} = \frac{\partial E}{\partial T} = \begin{cases} 0 & t > 0 \\ -1/8u & t < 0. \end{cases}$$

(b) After substitution into the Ginzburg-Landau Hamiltonian, a quadratic expansion of the free energy functional

$$\begin{aligned} (\nabla \mathbf{m})^2 &= (\nabla \phi_l)^2 + (\nabla \phi_t)^2, \\ \mathbf{m}^2 &= \bar{m}^2 + 2\bar{m}\phi_l + \phi_l^2 + \phi_t^2, \\ \mathbf{m}^4 &= \bar{m}^4 + 4\bar{m}^3\phi_l + 6\bar{m}^2\phi_l^2 + 2\bar{m}^2\phi_t^2 + O(\phi_l^3, \phi_l\phi_t^2), \end{aligned}$$

generates the perturbative expansion of the Hamiltonian

$$\begin{aligned} \beta H &= V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \int d\mathbf{x}\, \left[\frac{K}{2}(\nabla\phi_l)^2 + \frac{t + 12u\bar{m}^2}{2}\phi_l^2\right] \\ &\quad + \int d\mathbf{x}\, \left[\frac{K}{2}(\nabla\phi_t)^2 + \frac{t + 4u\bar{m}^2}{2}\phi_t^2\right] + O(\phi_l^3, \phi_l\phi_t^2). \end{aligned}$$

(c) Expressed in the Fourier basis, the quadratic Hamiltonian is diagonal. The correlation function

$$\langle \phi_\alpha(\mathbf{q})\phi_\beta(\mathbf{q}')\rangle = \delta_{\alpha\beta}(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')G_\alpha(\mathbf{q}), \qquad G_\alpha^{-1}(\mathbf{q}) = K(\mathbf{q}^2 + \xi_\alpha^{-2})$$

where

$$\begin{aligned} \frac{K}{\xi_l^2} &\equiv t + 12u\bar{m}^2 = \begin{cases} t & t > 0, \\ -2t & t < 0, \end{cases} \\ \frac{K}{\xi_t^2} &\equiv t + 4u\bar{m}^2 = \begin{cases} t & t > 0, \\ 0 & t < 0. \end{cases} \end{aligned}$$

Then, expressed in real space, one obtains the correlation function

$$\begin{aligned} G^c_{\alpha,\beta}(\mathbf{x},\mathbf{x}') &\equiv \langle (m_\alpha(\mathbf{x}) - \bar{m}_\alpha)(m_\beta(\mathbf{x}') - \bar{m}_\beta)\rangle = \langle \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{x}')\rangle, \\ &= -\frac{\delta_{\alpha\beta}}{K} I_d(\mathbf{x}-\mathbf{x}', \xi_\alpha), \end{aligned} \tag{6.3}$$

where

$$I_d(\mathbf{x},\xi) = -\int \frac{d\mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \xi^{-2}}.$$

Using this result and the expression given in the text, one obtains the asymptotics of the distribution function.

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26. The divergence of the correlation length at a second order phase transition implies self-similarity of spatial correlations. This, in turn, implies that the form of the free energy remains invariant under coordinate rescaling. This invariance is exploited in the renormalisation group procedure: The scaling of the parameters of the Ginzburg-Landau Hamiltonian under coordinate rescaling allows an identification of the fixed theory and the exposes the nature of the critical point. Operationally, the renormalisation procedure is implemented in three steps described in detail in the question:

    (a) Expressed in a Fourier representation

    $$\mathbf{m}(\mathbf{x}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \mathbf{m}(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}}$$

    With this definition, the long-range coupling of the Hamiltonian takes the form

    $$\frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d} J(\mathbf{q})\mathbf{m}(\mathbf{q})\cdot\mathbf{m}(-\mathbf{q}), \qquad J(\mathbf{q}) = \int d^d\mathbf{x} J(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}} = K_\sigma |\mathbf{q}|^\sigma$$

    With this result we obtain the expression shown in the text.

    (b) Coarse-Grain: The first step of the RG involves the elimination of fluctuations at scales $a < |\mathbf{x}| < ba$ or Fourier modes with wavevectors $\Lambda/b < |\mathbf{q}| < \Lambda$. Applied to the Gaussian model described in the text, the fields can be separated into slowly and rapidly varying functions, $\mathbf{m}(\mathbf{q}) = \mathbf{m}_>(\mathbf{q}) + \mathbf{m}_<(\mathbf{q})$, where

    $$\mathbf{m}(\mathbf{q}) = \begin{cases} \mathbf{m}_<(\mathbf{q}) & 0 < |\mathbf{q}| < \Lambda/b, \\ \mathbf{m}_>(\mathbf{q}) & \Lambda/b < |\mathbf{q}| < \Lambda. \end{cases}$$

    Since the Ginzburg-Landau functional is Gaussian, the partition function is separable in the modes and can be reexpressed in the form

    $$\mathcal{Z} = \int D\mathbf{m}_<(\mathbf{q}) e^{-\beta H[\mathbf{m}_<]} \int D\mathbf{m}_>(\mathbf{q}) e^{-\beta H[\mathbf{m}_>]}.$$

More precisely, the latter takes the form

$$\mathcal{Z} = \mathcal{Z}_> \int D\mathbf{m}_<(\mathbf{q}) \exp\left[-\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{t + K_2\mathbf{q}^2 + K_\sigma|\mathbf{q}|^\sigma}{2}\right)|\mathbf{m}_<(\mathbf{q})|^2\right],$$

where $\mathcal{Z}_>$ represents some irrelevant constant.

Rescale: The partition function for the modes $\mathbf{m}_<(\mathbf{q})$ is similar to the original, except that the upper cut-off has decreased to Λ/b, reflecting the coarse-graining in resolution. The rescaling, $\mathbf{x}' = \mathbf{x}/b$ in real space, is equivalent to $\mathbf{q}' = b\mathbf{q}$ in momentum space, and restores the cut-off to the original value.

Renormalise: The final step of the RG is the renormalisation of the field, $\mathbf{m}'(\mathbf{x}') = \mathbf{m}_<(\mathbf{x}')/\zeta$. Alternatively, we can renormalise the Fourier modes according to $\mathbf{m}'(\mathbf{q}') = \mathbf{m}_<(\mathbf{q}')/z$, resulting in

$$\begin{aligned}
\mathcal{Z} &= \mathcal{Z}_> \int D\mathbf{m}'(\mathbf{q}')e^{-\beta H'[\mathbf{m}'(\mathbf{q}')]},\\
\beta H' &= \int_0^\Lambda \frac{d^d\mathbf{q}'}{(2\pi)^d} b^{-d}z^2\left(\frac{t + K_2 b^{-2}\mathbf{q}'^2 + K_\sigma b^{-\sigma}|\mathbf{q}'|^\sigma}{2}\right)|\mathbf{m}'(\mathbf{q}')|^2.
\end{aligned}$$

As a result of the RG procedure the set of parameters $\{t, K_2, K_\sigma\}$ has transformed from to a new set

$$\begin{cases} t' = tb^{-d}z^2,\\ K_2' = K_2 b^{-d-2}z^2,\\ K_\sigma' = K_\sigma b^{-d-\sigma}z^2.\end{cases}$$

Setting $K_2' = K_2$, the fluctuations are made scale invariant by the choice $z = b^{1+d/2}$ from which one obtains the scaling relations

$$\begin{cases} t' = b^2 t & y_t = 2,\\ K_\sigma' = K_\sigma b^{2-\sigma} & y_\sigma = 2-\sigma.\end{cases}$$

Thus for $\sigma > 2$, the parameter K_σ scales to zero. In this case the fixed Hamiltonian is simply

$$\beta H^* = \int d^d\mathbf{x}\frac{K_2}{2}(\nabla\mathbf{m})^2$$

(b) For $\sigma < 2$ setting $K_\sigma' = K_\sigma$, $z = (\sigma + d)/2$ and one obtains

$$\begin{cases} t' = b^\sigma t & y_t = \sigma,\\ K_2' = K_2 b^{\sigma-2} & y_2 = \sigma - 2.\end{cases}$$

In this case K_2 scales to zero and the fixed Hamiltonian takes the form

$$\beta H^* = \frac{1}{2}\int d^d\mathbf{x}_1 \int d^d\mathbf{x}_2 J(\mathbf{x}_1 - \mathbf{x}_2)\mathbf{m}(\mathbf{x}_1)\cdot\mathbf{m}(\mathbf{x}_2)$$

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27. Essay question: Refer to Lecture notes

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28. According to the Lagrangian formulation of the Feynman path integral one has

$$\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \int_{q(0)=q_I, q(t)=q_F} Dq e^{iS/\hbar},$$

where S denotes the classical action. For the harmonic oscillator, the classical action is given by

$$S = \int_0^t dt' \left(\frac{m}{2}\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \right)$$

(a) The evaluation of the path integral is most readily performed by dividing the functional integral into a contribution arising from the classical path and fluctuations around the classical path. With the boundary conditions as specified in the question, the classical path is parameterised by the trajectory

$$q(t') = A\cos\omega t + B\sin\omega t$$

where

$$A = q, \qquad B = q\frac{1-\cos\omega t}{\sin\omega t}$$

Evaluating the classical action, one obtains

$$S = m\omega^2 \left[(A^2 - B^2)\frac{\sin 2\omega t}{4\omega} + 2AB\frac{\sin^2 \omega t}{2\omega} \right]$$

Substituting for A and B, after some algebra, one obtains the exponent shown in the question.
To evaluate the contribution from the fluctuations, one must evaluate the part integral with the boundary conditions $q(t) = q(0) = 0$. Being Gaussian in the fields q, the integral may be performed and one obtains

$$\langle 0 | e^{-i\hat{H}t/\hbar} | 0 \rangle = J \det\left(-\partial_t^2 - \omega^2 \right)^{-1/2}$$

where J denotes some unknown constant. However, the propagator may be normalised by the free particle propagator. Substituting for the eigenvalues of the operator, and normalising, one obtains

$$\langle 0 | e^{-i\hat{H}t/\hbar} | 0 \rangle = \prod_{n>0} \left(1 - \left(\frac{\omega t}{n\pi} \right)^2 \right)^{-1/2} G_{\text{free}}(0,0).$$

Making use of the formula given in the question together with the free particle propagator, one obtains the required result.

(b) The evaluation of the quantum partition function can be made by use of the Feynman path integral. Making use of the identity

$$\mathcal{Z} = \int_{-\infty}^{\infty} dq \langle q | e^{-\beta \hat{H}} | q \rangle$$

together with the path integral above with $it/\hbar = \beta$, one obtains

$$\mathcal{Z} = \frac{1}{2\sinh(\beta\hbar\omega/2)}.$$

This results is easily confirmed by direct evaluation of the sum

$$\mathcal{Z} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}$$

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29. (a) In the mean-field approximation, the free energy of a system close to a second order phase transition can be shown to take a homogeneous form. According to the scaling hypothesis, although mean-field theory becomes invalid below the upper critical dimension, the homogeneous form of the free energy (and therefore of any other thermodynamic quantity) is maintained. More precisely

$$f_{\text{sing.}}(t, h) = t^{2-\alpha} g_f(h/t^{\Delta})$$

where the actual exponents α and Δ depend on the critical point being considered.

With this assumption, the critical exponents can be shown to be connected by exponent identities:

(a) For example, from the free energy, one obtains the magnetisation as

$$m(t, h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^{\Delta}).$$

In the limit $x \to 0$, $g_m(x)$ is a constant, and $m(t, h = 0) \sim t^{2-\alpha-\Delta}$ (i.e. $\beta = 2 - \alpha - \Delta$). On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and $m(t = 0, h) \sim t^{2-\alpha-\Delta}(h/t^{\Delta})^p$. Since this limit is independent of t, we must have $p\Delta = 2-\alpha-\Delta$. Hence $m(t = 0, h) \sim h^{(2-\alpha-\Delta)/\Delta}$ (i.e. $\delta = \Delta/(2-\alpha-\Delta) = \Delta/\beta$).

(b) From the magnetisation, one obtains the susceptibility

$$\chi(t, h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^{\Delta}) \Rightarrow \chi(t, h = 0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha.$$

Now close to criticality, the correlation length ξ is solely responsible for singular contributions to thermodynamic quantities. Since $\ln \mathcal{Z}(t, h)$ is dimensionless and extensive (i.e. $\propto L^d$), it must take the form

$$\ln \mathcal{Z} = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). (A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$. The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t,h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^{\Delta})$$

As a consequence, comparing with the homogeneous expression for the free energy, one obtains the Josephson identity

$$2 - \alpha = d\nu$$

(b) According to the Mermin-Wagner theorem, spontaneous symmetry breaking of a continuous symmetry leads to the appearance of Goldstone modes which destroy long-range order in dimensions $d \leq 2$. However, in two-dimensions, there exists a low temperature phase of quasi long-range order in which the correlations decay algebraically at long-distances. This leaves open the room for a phase transition at some intermediate temperature in which the correlation function crosses over to exponential decay.

To understand the nature of the transition, it is necessary to take into account the existence of topological defects, vortex configurations of the fields. The elementary defect which has a unit charge involves a 2π twist of θ as one encircles the defect. More formally,

$$\oint \nabla\theta \cdot d\ell = 2\pi n \quad \Longrightarrow \quad \nabla\theta = \frac{n}{r}\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z,$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_z$ are unit vectors respectively in the plane and perpendicular to it. This (continuum) approximation fails close to the centre (core) of the vortex, where the lattice structure is important.

The energy cost from a single vortex of charge n has contributions from the core region, as well as from the relatively uniform distortions away from the centre. The distinction between regions inside and outside the core is arbitrary, and for simplicity, we shall use a circle of radius a to distinguish the two, i.e.

$$\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a d^2\mathbf{x}(\nabla\theta)^2 = \beta E_n^0(a) + \pi K n^2 \ln\left(\frac{L}{a}\right).$$

The dominant part of the energy comes from the region outside the core and diverges logarithmically with the system size L. The large energy cost associated with the defects prevents their spontaneous formation close to zero temperature. The partition function for a configuration with a single vortex of charge n is

$$\mathcal{Z}_1(n) \approx \left(\frac{L}{a}\right)^2 \exp\left[-\beta E_n^0(a) - \pi K n^2 \ln\left(\frac{L}{a}\right)\right],$$

where the factor of $(L/a)^2$ results from the configurational entropy of possible vortex locations in an area of size L^2. The entropy and energy of a vortex both grow as $\ln L$, and the free energy is dominated by one or the other. At low temperatures, large K, energy dominates and $\mathcal{Z}_1$, a measure of the weight of configurations with a single vortex, vanishes. At high enough temperatures, $K < K_n = 2/(\pi n^2)$, the entropy contribution is large enough to favour spontaneous formation of vortices. On increasing temperature, the first vortices that appear correspond to $n = \pm 1$ at $K_c = 2/\pi$. Beyond this point many vortices appear and the equation above is no longer applicable.

In fact this estimate of K_c represents only a lower bound for the stability of the system towards the condensation of topological defects. This is because pairs (dipoles) of defects may appear at larger couplings. Consider a pair of charges ± 1 separated by a distance d. Distortions far from the core $|\mathbf{r}| \gg d$ can be obtained by superposing those of the individual vortices

$$\nabla\theta = \nabla\theta_+ + \nabla\theta_- \approx 2\mathbf{d} \cdot \nabla \left(\frac{\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z}{|\mathbf{r}|} \right),$$

which decays as $d/|\mathbf{r}|^2$. Integrating this distortion leads to a finite energy, and hence dipoles appear with the appropriate Boltzmann weight at any temperature. The low temperature phase should therefore be visualised as a gas of tightly bound dipoles, their density and size increasing with temperature. The high temperature phase constitutes a plasma of unbound vortices.

(c) Starting from the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$$

the time evolution operator is defined by

$$|\Psi(t')\rangle = \hat{U}(t', t)|\Psi(t)\rangle, \qquad \hat{U}(t', t) = \exp\left[-\frac{i}{\hbar}\hat{H}(t' - t)\right].$$

In the real space representation

$$U(x', t'; x, t) = \langle x'| \exp\left[-\frac{i}{\hbar}\hat{H}(t' - t)\right] |x\rangle,$$

According to the Feynman path integral, the quantum evolution operator is expressed as the sum over all trajectories subject to the boundary conditions and weighted by the classical action. In the Hamiltonian formulation,

$$U(x', t'; x, t) = \int Dx(t) \int Dp(t) \exp\left[\frac{i}{\hbar}S(p, x)\right],$$

$$S(p, x) = \int_t^{t'} dt'' \left[p\dot{x} - H(p, x)\right],$$

and in the Lagrangian formulation,

$$U(x',t';x,t) = \int \bar{D}x(t)\exp\left[\frac{i}{\hbar}S(x)\right],$$

$$S(x) = \int_t^{t'} dt''\left[\frac{m}{2}\dot{x}^2 - V(x)\right].$$

To establish an analogy with statistical mechanics we have to consider propagation in *imaginary* or *Euclidean time* T. In this way, we obtain

$$U(x',t'=-iT;x,t=0) = \int Dx(\tau)\exp\left[-\frac{1}{\hbar}S(x)\right],$$

where

$$S(x) = \int_0^T d\tau\left[\frac{m}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + V(x(\tau))\right].$$

Interpreting the action as a classical free energy functional, and the path integral as a classical partition function, one has the analogy: The transition amplitude for a quantum particle for the time (-iT) is equal to the classical partition function for a string of length T computed at the value $\beta = 1/\hbar$.
A second analogy follows from the fact that the *quantum* partition function for the particle is given by $\mathcal{Z}_{\rm qu} = {\rm Tr}\exp[-\beta H]$ and hence,

$$\mathcal{Z}_{\rm qu} = \int dx\langle x|e^{-\beta H}|x\rangle = \int dxU(x,t'=-i\beta\hbar;x,t=0).$$

Therefore, in quantum statistical mechanics, the inverse temperature plays the role of an imaginary time.

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30. A second order phase transition is associated with the continuous development of an order parameter. At the critical point, various correlation functions are seen to exhibit singular behaviour. In particular, the correlation length, the typical scale over which fluctuations are correlated, diverges. This fact motivates the consideration of a coarse-grained phenomenology in which the microscopic details of the system are surrendered.

    To this end, one introduces a coarse-grained order parameter which varies on a length scale much greater than the microscopic scales. The effective Hamiltonian, is phenomenological depending only on the fundamental symmetries of the system.

    (a) In the Landau mean-field approximation the partition function

$$\mathcal{Z} = \int Dme^{-\beta H[m]} \sim^{\rm max}_{m} e^{-\beta H[m]}$$

In this approximation, the integrand is maximised when the field configuration $m(\mathbf{x})$ is homogeneous and minimises the dimensionless free energy density

$$f = \frac{\beta F}{V} = \frac{t}{2}m^2 + um^4 - hm.$$

Varying f with respect to m and setting $h = 0$, one finds the solution

$$\bar{m} = \begin{cases} \sqrt{\frac{-t}{4u}} & t < 0 \\ 0 & t > 0. \end{cases}$$

Varying $\partial f/\partial m|_{m=\bar{m}}$ with respect to $\bar{m}$ one obtains

$$t + 12u\bar{m}^2 - \frac{\partial h}{\partial \bar{m}} = 0$$

From this result we obtain the susceptibility

$$\chi^{-1} = \frac{\partial h}{\partial \bar{m}}\bigg|_{h=0} = t + 12u\bar{m}^2 = \begin{cases} -2t & t < 0 \\ t & t > 0. \end{cases}$$

(b) Expanding in fluctuations around the mean-field, one finds

$$\int d^d\mathbf{x}\left(\frac{t}{2}m^2 + um^4\right) = Vf[\bar{m}] + \int d^d\mathbf{x}\frac{K\xi^{-2}}{2}\phi^2$$

where

$$\xi^{-2} = \begin{cases} -2t/K & t < 0 \\ t/K & t > 0. \end{cases}$$

Taken together with the gradient term, we obtain the Hamiltonian as specified.

(c) In the Fourier representation, we have

$$\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle = (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\frac{1}{K(\mathbf{q}^2 + \xi^{-2})}$$

Using the result, one finds that

$$\langle m(\mathbf{x})m(0)\rangle - \bar{m}^2 = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{K(\mathbf{q}^2 + \xi^{-2})} \sim e^{-|\mathbf{x}|/\xi}.$$

From this result we learn that ξ can be interpreted as the correlation length.

---

31. The divergence of the correlation length at a second order phase transition suggests that in the vicinity of the transition, the microscopic lengths are irrelevant. The critical behaviour is dominated by fluctuations that are statistically self-similar up to the scale ξ. Self-similarity allows the gradual elimination of the correlated degrees of freedom at length scales $x \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at scale ξ.

(a) This is achieved through a procedure known as the Renormalisation Group (RG), whose conceptual foundation is outlined in the three steps below:

Coarse-Grain: eliminate fluctuations at scales $a < |\mathbf{x}| < ba$. Since the Ginzburg-Landau functional is Gaussian, by setting

$$\mathbf{m}(\mathbf{q}) = \begin{cases} \mathbf{m}_<(\mathbf{q}) & 0 < |\mathbf{q}| < \Lambda/b \\ \mathbf{m}_>(\mathbf{q}) & \Lambda/b < |\mathbf{q}| < \Lambda \end{cases}$$

the partition function is separable in the modes and can be reexpressed in the form

$$\mathcal{Z} = \mathcal{Z}_> \int D\mathbf{m}_<(\mathbf{q}) \exp\left[-\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + K\mathbf{q}^2}{2} \right) |\mathbf{m}_<(\mathbf{q})|^2 + \mathbf{h}\cdot\mathbf{m}_<(0) \right],$$

where $\mathcal{Z}_> = \exp[-(nV/2)\int_{\Lambda/b}^{\Lambda}(d^d\mathbf{q}/(2\pi)^d)\ln(t + K\mathbf{q}^2)]$. [Full credit does not require an evaluation of the functional integral over $\mathbf{m}_>$.]

Rescale: the partition function for the modes $\mathbf{m}_<(\mathbf{q})$ is similar to the original, except that the upper cut-off has decreased to Λ/b, reflecting the coarse-graining in resolution. The rescaling, $\mathbf{x}' = \mathbf{x}/b$ in real space, is equivalent to $\mathbf{q}' = b\mathbf{q}$ in momentum space, and restores the cut-off to the original value.

Renormalise: the final step of the RG is the renormalisation of the field, $\mathbf{m}'(\mathbf{x}') = \mathbf{m}_<(\mathbf{x}')/\zeta$. Alternatively, we can renormalise the Fourier modes according to $\mathbf{m}'(\mathbf{q}') = \mathbf{m}_<(\mathbf{q}')/z$, resulting in

$$\begin{aligned} \mathcal{Z} &= \mathcal{Z}_> \int D\mathbf{m}'(\mathbf{q}') e^{-\beta H'[\mathbf{m}'(\mathbf{q}')]}, \\ \beta H'[\mathbf{m}'] &= \int_0^{\Lambda} \frac{d^d\mathbf{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{t + Kb^{-2}\mathbf{q}'^2}{2} \right) |\mathbf{m}'(\mathbf{q}')|^2 - z\mathbf{h}\cdot\mathbf{m}'(0). \end{aligned}$$

As a result of the RG procedure the set of parameters $\{K, t, h\}$ has transformed from to a new set

$$\begin{cases} K' = Kb^{-d-2}z^2, \\ t' = tb^{-d}z^2, \\ h' = hz. \end{cases}$$

(b) The point $t = h = 0$ is fixed and represents the Gaussian fixed point. To make the fluctuations scale invariant at this point, we must ensure that the remaining parameter in the Hamiltonian K also stays fixed. This is achieved by the choice $z = b^{1+d/2}$ which implies

$$\begin{cases} t' = b^2 t & y_t = 2, \\ h' = b^{1+d/2} h & y_h = 1 + d/2. \end{cases}$$

From these equations, we can predict the scaling of the Free energy

$$\begin{aligned} f_{\text{sing.}}(t, h) &= b^{-d} f_{\text{sing.}}(b^2 t, b^{1+d/2} h), \qquad b^2 t = 1, \\ &= t^{d/2} g_f(h/t^{1/2+d/4}). \end{aligned}$$

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32. In a second order phase transition an order parameter grows continuously from zero. The onset of order below the transition is accompanied by a spontaneous symmetry breaking — the symmetry of the low temperature ordered phase is lower than the symmetry of the high temperature disordered phase. An example is provided by the classical ferromagnet where the appearance of net magnetisation breaks the symmetry $m \mapsto -m$. If the symmetry is continuous, the spontaneous breaking of symmetry is accompanied by the appearance of massless Goldstone mode excitations. In the magnet, these excitations are known as spin waves.

(a) Applying the rules of Gaussian functional integration, one finds that $\langle\theta(\mathbf{x})\rangle = 0$, and the correlation function takes the form

$$G(\mathbf{x},\mathbf{x}') \equiv \langle\theta(\mathbf{x})\theta(0)\rangle = -\frac{C_d(\mathbf{x})}{\bar{K}}, \qquad \nabla^2 C_d(\mathbf{x}) = \delta^d(\mathbf{x})$$

where C_d denotes the Coulomb potential for a δ-function charge distribution. Exploiting the symmetry of the field, and employing Gauss', $\int d\mathbf{x}\ \nabla^2 C_d(\mathbf{x}) = \oint dS \cdot \nabla C_d$, one finds that C_d depends only on the radial coordinate x, and

$$\frac{dC_d}{dx} = \frac{1}{x^{d-1}S_d}, \qquad C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + \text{const.},$$

where $S_d = 2\pi^{d/2}/(d/2-1)!$ denotes the total d-dimensional solid angle.

(b) Using this result, one finds that

$$\left\langle[\theta(\mathbf{x}) - \theta(0)]^2\right\rangle = 2\left[\langle\theta(0)^2\rangle - \langle\theta(\mathbf{x})\theta(0)\rangle\right] \overset{|\mathbf{x}|>a}{=} \frac{2(|\mathbf{x}|^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d},$$

where the cut-off, a is of the order of the lattice spacing. (Note that the case where $d = 2$, the combination $|\mathbf{x}|^{2-d}/(2-d)$ must be interpreted as $\ln|\mathbf{x}|$.

This result shows that the long distance behaviour changes dramatically at $d = 2$. For $d > 2$, the phase fluctuations approach some finite constant as $|\mathbf{x}| \to \infty$, while they become asymptotically large for $d \leq 2$. Since the phase is bounded by 2π, it implies that long-range order (predicted by the mean-field theory) is destroyed.

Turning to the two-point correlation function of $\mathbf{m}$, and making use of the Gaussian functional integral, obtains

$$\langle\mathbf{m}(\mathbf{x})\cdot\mathbf{m}(0)\rangle = \bar{m}^2 \text{Re}\ \left\langle e^{i[\theta(\mathbf{x})-\theta(0)]}\right\rangle.$$

For Gaussian distributed variables $\langle\exp[\alpha\theta]\rangle = \exp[\alpha^2\langle\theta^2\rangle/2]$. We thus obtain

$$\langle\mathbf{m}(\mathbf{x})\cdot\mathbf{m}(0)\rangle = \bar{m}^2\exp\left[-\frac{1}{2}\langle[\theta(\mathbf{x})-\theta(0)]^2\rangle\right] = \bar{m}^2\exp\left[-\frac{(|\mathbf{x}|^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d}\right],$$

implying a power-law decay of correlations in $d = 2$, and an exponential decay in $d < 2$. From this result we find

$$\lim_{|\mathbf{x}|\to\infty} \langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0) \rangle = \begin{cases} m_0^2 & d > 2, \\ 0 & d \le 2. \end{cases}$$

(c) In the context of the XY-spin model, the perturbative analysis above represents a spin wave expansion of the Hamiltonian. However, the spontaneous breaking of the continuous symmetry admits the existence of topological non-trivial field configurations which connect the degenerate minima. In the present context, these non-perturbative configurations are vortices.

Now it is easy to show that the energy of a single vortex diverges logarithmically with the system size. At the same time, the vortex recovers a contribution to the free energy from the entropy which also diverges logarithmically with the system size. The trade between these factors leads to a catastrophic condensation of vortices at some finite temperature which separates a phase of quasi-LRO with power law correlations from a disordered phase — the Berezinskii-Kosterlitz-Thouless transition.

As a matter of detail, it should be noted that the low temperature phase is more precisely one in which pairs of oppositely charged vortices are bound in dipole pairs whereas the high temperature phase is a free plasma of vortices.

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33. In mean-field, the free energy can be shown to take a homogeneous form around a second order transition. According to the scaling hypothesis, when one goes beyond mean-field, homogeneity of the singular form of the free energy (and of any other thermodynamic quantity) retains the homogeneous form

$$f_{\text{sing.}}(t,h) = t^{2-\alpha} g_f(h/t^{\Delta})$$

where the actual exponents α and Δ depend on the critical point being considered.

(a) From the free energy, one obtains the magnetisation as

$$m(t,h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^{\Delta}).$$

In the limit $x \to 0$, $g_m(x)$ is a constant, and $m(t,h=0) \sim t^{2-\alpha-\Delta}$ (i.e. $\beta = 2-\alpha-\Delta$). On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and $m(t=0,h) \sim t^{2-\alpha-\Delta}(h/t^{\Delta})^p$. Since this limit is independent of t, we must have $p\Delta = 2-\alpha-\Delta$. Hence $m(t=0,h) \sim h^{(2-\alpha-\Delta)/\Delta}$ (i.e. $\delta = \Delta/(2-\alpha-\Delta) = \Delta/\beta$).

(b) From the magnetisation, one obtains the susceptibility

$$\chi(t,h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^{\Delta}) \Rightarrow \chi(t,h=0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha.$$

(c) Close to criticality, the correlation length ξ is solely responsible for singular contributions to thermodynamic quantities. Since $\ln \mathcal{Z}(t,h)$ is dimensionless and extensive (i.e. $\propto L^d$), it must take the form

$$\ln \mathcal{Z} = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). (A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$. The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t,h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^{\Delta})$$

As a consequence, comparing with the homogeneous expression for the free energy, one obtains the Josephson identity

$$2 - \alpha = d\nu$$

(d) Using the correlation function one obtains the susceptibility

$$\begin{aligned} \chi(t,h) &\sim \int d\mathbf{x}\ \langle m(\mathbf{x}) m(0)\rangle \\ &\sim \int_0^{\xi} dx \frac{x^{d-1}}{x^{d-2+\eta}} \sim \xi^{2-\eta} \\ &\sim t^{-(2-\eta)\nu} g_\xi\left(\frac{h}{t^{\Delta}}\right) \end{aligned}$$

We thus obtain the exponent identity $\gamma = (2-\eta)\nu$.

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34. (a) essay on Ginzburg-Landau theory

(b) essay on Ginzburg Criterion

(c) The divergence of the correlation length at a second order phase transition suggests that in the vicinity of the transition, the microscopic lengths are irrelevant. The critical behaviour is dominated by fluctuations that are statistically self-similar up to the scale ξ. Self-similarity allows the gradual elimination of the correlated degrees of freedom at length scales $x \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at scale ξ. This is achieved through a procedure known as the Renormalisation Group (RG), whose conceptual foundation is outlined below:

(i) *Coarse-Grain*: The first step of the RG is to decrease the resolution by changing the minimum length scale from the microscopic scale a to ba where $b > 1$. The coarse-grained magnetisation is then

$$\bar{\mathbf{m}}(\mathbf{x}) = \frac{1}{b^d} \int_{\text{Cell centred at } \mathbf{x}} d\mathbf{y}\ \mathbf{m}(\mathbf{y}).$$

(ii) *Rescale*: Due to the change in resolution, the coarse-grained "picture" is grainier than the original. The original resolution a can be restored by decreasing all length scales by a factor b, i.e. defining

$$\mathbf{x}' = \frac{\mathbf{x}}{b}.$$

Thus, at each position $\mathbf{x}'$ we have defined an average moment $\bar{\mathbf{m}}(\mathbf{x}')$.

(iii) *Renormalise*: The relative size of the fluctuations of the rescaled magnetisation profile is in general different from the original, i.e. there is a change in contrast between the pictures. This can be remedied by introducing a factor ζ, and defining a renormalised magnetisation

$$\mathbf{m}'(\mathbf{x}') = \frac{1}{\zeta}\bar{\mathbf{m}}(\mathbf{x}').$$

The choice of ζ will be discussed later.

By following these steps, for each configuration $\mathbf{m}(\mathbf{x})$ we generate a renormalised configuration $\mathbf{m}'(\mathbf{x}')$. It can be regarded as a mapping of one set of random variables to another, and can be used to construct the probability distribution. Kadanoff's insight was to realise that since, on length scales less than ξ, the renormalised configurations are statistically similar to the original ones, they must be distributed by a Hamiltonian that is also close to the original. In particular, if the original Hamiltonian βH is at a critical point, $t = h = 0$, the new $(\beta H)'$ is also at criticality since no new length scale is generated in the renormalisation procedure, i.e. $t' = h' = 0$.

However, if the Hamiltonian is originally off criticality, then the renormalisation takes us further away from criticality because $\xi' = \xi/b$ is smaller. The next assumption is that since any transformation only involves changes at the shortest length scales it can not produce singularities. The renormalised parameters must be *analytic* functions, and hence expandable as

$$\begin{cases} t(b;t,h) = A(b)t + B(b)h + O(t^2, h^2, th), \\ h(b;t,h) = C(b)t + D(b)h + O(t^2, h^2, th). \end{cases}$$

However, the known behaviour at $t = h = 0$ rules out a constant term in the expansion, and to prevent a spontaneously broken symmetry we further require $C(b) = B(b) = 0$. Finally, commutativity $A(b_1 \times b_2) = A(b_1) \times A(b_2)$ implies $A(b) = b^{y_t}$ and $D(b) = b^{y_h}$. So, to lowest order

$$\begin{cases} t_b \equiv t(b) = b^{y_t} t, \\ h_b \equiv h(b) = b^{y_h} h. \end{cases}$$

where y_t, $y_h > 0$. As a consequence, since the statistical weight of new configuration, $W'[\mathbf{m}']$ is the sum of the weights $W[\mathbf{m}]$ of old ones, the partition function is preserved

$$\mathcal{Z} = \int D\mathbf{m}\ W[\mathbf{m}] = \int D\mathbf{m}'\ W'[\mathbf{m}'] = \mathcal{Z}'.$$

From this is follows that the free energies density takes the form

$$f(t,h) = -\frac{\ln \mathcal{Z}}{V} = -\frac{\ln \mathcal{Z}'}{V' b^d} = b^{-d} f(t_b, h_b) = b^{-d} f(b^{y_t} t, b^{y_h} h),$$

where we have assumed that the two free energies are obtained from the *same Hamiltonian* in which only the parameters t and h have changed. The free energy describes a homogeneous function of t and h. This is made apparent by choosing a rescaling factor b such that $b^{y_t} t$ is a constant, say unity, i.e. $b = t^{-1/y_t}$, and

$$f(t,h) = t^{d/y_t} f(1, h/t^{y_h/y_t}) \equiv t^{d/y_t} g_f(h/t^{y_h/y_t})$$

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35. (a) Applying the Hubbard-Stratonovich transformation,

$$\exp\left[\sum_{ij} J_{ij}\sigma_i\sigma_j\right] = C\int \prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i [J^{-1}]_{ij} m_j + 2\sum_i \sigma_i m_i\right]$$

the classical partition function $\mathcal{Z} = \sum_{\{\sigma_i\}} e^{-\beta H[\sigma_i]}$ is given by

$$\mathcal{Z} \ = \ C\int \prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i [J^{-1}]_{ij} m_j + \sum_i \ln\left(2\cosh(2m_i + h)\right)\right].$$

(b) To determine $[J^{-1}]_{ij}$, we transform to Fourier space. In particular, for the model at hand, the eigenvalues of J_{ij} are given by

$$J(q) = \sum_{n=-\infty}^{\infty} e^{iqn} J e^{-\kappa|n|} = \frac{J}{c - b\cos q}$$

where $c = \coth\kappa$ and $b = 1/\sinh\kappa$. Making use of this result we obtain

$$\begin{aligned}[J^{-1}]_{ij} &= \int_{-\pi}^{\pi} \frac{dq}{2\pi}\frac{e^{-iq(n_i - n_j)}}{J(q)} \\ &= \frac{1}{J}\begin{bmatrix} c & -b/2 & & & \\ -b/2 & c & -b/2 & & \\ & -b/2 & c & -b/2 & \\ & & -b/2 & c & -b/2 \\ & & & -b/2 & c \end{bmatrix}\end{aligned}$$

Therefore

$$\mathcal{Z} = C \int \prod_k dm_k \exp\left[-\frac{b}{2J} \sum_i (m_i - m_{i+1})^2 - \sum_i U(m_i) \right]$$

where $U(m) = (c-b)m^2/J - \ln[2\cosh(2m+h)]$. In particular $c-b = \tanh(\kappa/2)$.

(c) For small m and h the effective free energy can be expanded as

$$U(m) = -\ln 2 + \frac{t}{2}m^2 + \frac{4}{3}m^4 - 2hm + \cdots$$

where $t/2 = \tanh(\kappa/2)/J - 2$. Evidently, at zero magnetic field, the effective potential $U(m)$ is quartic. For $t < 0$, the potential takes the form of a double well.

The path integral for a particle in a potential well is given by

$$\begin{aligned} \mathcal{Z} &= \int Dr(t) \exp\left[\frac{i}{\hbar} \int_0^t dt' \left(\frac{m}{2}\dot{r}^2 - U(r) \right) \right] \\ &= \int Dr(\tau) \exp\left[-\frac{1}{\hbar} \int_0^T d\tau' \left(\frac{m}{2}\dot{r}^2 + U(r) \right) \right] \end{aligned}$$

By identifying r with m, and τ with x, the partition function of the Ising model is seen to be equivalent to the path integral of a particle in a double well potential where the transition time T is equalent to the length of the spin chain L. From this analogy, the magnetic field can be seen as inducing an asymmetry of the potential.

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36. The upper critical dimension represents the dimensionality at and above which mean-field theory furnishes an accurate description of the behaviour close to the critical point.

(a) In the Landau mean-field approximation, the Free energy is assumed to be dominated by the field configuration with the minimum Hamiltonian. Since $K > 0$, this occurs when m is constant, homogeneous in space. In this case

$$f = \frac{\beta F}{V} = \min_m \left[\frac{t}{2}m^2 + \frac{v}{6}m^6 \right]$$

Minimising, one finds $t\bar{m} + v\bar{m}^5 = 0$ from which one obtains the solution

$$\bar{m}(t) = \begin{cases} 0 & t > 0 \\ (-t/v)^{1/4} & t < 0 \end{cases}$$

Substituting backing to the free energy, one obtains

$$f = \begin{cases} 0 & t > 0 \\ \frac{1}{3}t(-t/v)^{1/2} & t < 0 \end{cases}$$

from which one finds the heat capacity

$$C = \frac{\partial^2 f}{\partial t^2} = \begin{cases} 0 & t > 0 \\ -\frac{1}{4}(1/tv)^{1/2} & t < 0 \end{cases}$$

(b) Setting $m(\mathbf{x}) = \bar{m} + \phi(\mathbf{x})$ and expanding to quadratic order in the fluctuations $\phi(\mathbf{x})$, one finds

$$\begin{aligned} m^2(\mathbf{x}) &= \bar{m}^2 + 2\bar{m}\phi(\mathbf{x}) + \phi^2(\mathbf{x}) \\ m^6(\mathbf{x}) &= \bar{m}^6 + 6\bar{m}^5\phi(\mathbf{x}) + 15\bar{m}^4\phi^2(\mathbf{x}) + \cdots \end{aligned}$$

Put back into the Hamiltonian, one obtains

$$\beta H[\phi(\mathbf{x})] = fV + \int \mathrm{d}^d\mathbf{x} \left[\overbrace{(t/2 + (15/6)v\bar{m}^4)}^{(K/2)\xi^{-2}} \phi^2 + \frac{K}{2}(\nabla\phi)^2 \right] + \cdots$$

where

$$\xi^{-2} = \begin{cases} t/K & t > 0 \\ -4t/K & t < 0 \end{cases}$$

To quadratic order, one obtains the correlation function

$$\langle m(\mathbf{x})m(0)\rangle = \bar{m}^2 + \langle\phi(\mathbf{x})\phi(0)\rangle$$

where

$$\langle\phi(\mathbf{x})\phi(0)\rangle = \frac{1}{K}\int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \xi^{-2}}$$

Using the result given in the question, one can see that the correlation function decays exponentially on length scales $|\mathbf{x}| \gg \xi$ giving ξ the interpretation of the correlation length.

(c) Using the result above, and integrating over ϕ, the fluctuation correction to the free energy is given by

$$\delta(\beta F) = \frac{1}{2}\int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \ln\left[K\mathbf{q}^2 + K\xi^{-2}\right]$$

From this result, differentiating with respect to t, one obtains the fluctuation correction to the specific heat of the form

$$\delta C = \frac{1}{2}\begin{cases} \displaystyle\int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{1}{(K\mathbf{q}^2 + t)^2} & t > 0 \\ \displaystyle 8\int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{1}{(K\mathbf{q}^2 - 4t)^2} & t < 0 \end{cases}$$

In $d < 4$, the fluctuation contribution to the specific heat diverges on approaching the critical point as $K^{-d/2}|t|^{d/2-2}$. The latter is seen to overwhelm the mean-field contribution in dimensions $d < d_u = 3$.

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37. (a) In mean-field, the free energy can be shown to take a homogeneous form around a second order transition. According to the scaling hypothesis, when one goes beyond mean-field, homogeneity of the singular form of the free energy (and of any other thermodynamic quantity) retains the homogeneous form

$$f_{\text{sing.}}(t,h) = t^{2-\alpha} g_f(h/t^{\Delta})$$

where the actual exponents α and Δ depend on the critical point being considered. (Additional credit for discussion of hyperscaling.)

(i) From the free energy, one obtains the magnetisation as

$$m(t,h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^{\Delta}).$$

In the limit $x \to 0$, $g_m(x)$ is a constant, and $m(t,h=0) \sim t^{2-\alpha-\Delta}$ (i.e. $\beta = 2-\alpha-\Delta$). On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and $m(t=0,h) \sim t^{2-\alpha-\Delta}(h/t^{\Delta})^p$. Since this limit is independent of t, we must have $p\Delta = 2-\alpha-\Delta$. Hence $m(t=0,h) \sim h^{(2-\alpha-\Delta)/\Delta}$ (i.e. $\delta = \Delta/(2-\alpha-\Delta) = \Delta/\beta$).

(ii) From the magnetisation, one obtains the susceptibility

$$\chi(t,h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^{\Delta}) \Rightarrow \chi(t,h=0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha.$$

(iii) Close to criticality, the correlation length ξ is solely responsible for singular contributions to thermodynamic quantities. Since $\ln \mathcal{Z}(t,h)$ is dimensionless and extensive (i.e. $\propto L^d$), it must take the form

$$\ln \mathcal{Z} = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). (A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$. The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t,h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^{\Delta})$$

As a consequence, comparing with the homogeneous expression for the free energy, one obtains the Josephson identity

$$2 - \alpha = d\nu$$

(b) According to the Mermin-Wagner theorem, spontaneous symmetry breaking of a continuous symmetry leads to the appearance of Goldstone modes which destroy long-range order in dimensions $d \leq 2$. However, in two-dimensions, there exists a low temperature phase of quasi long-range order in which the correlations decay algebraically at long-distances. This leaves open the room for a phase transition at some intermediate temperature in which the correlation function crosses over to exponential decay.

To understand the nature of the transition, it is necessary to take into account the existence of topological defects, vortex configurations of the fields. The elementary defect which has a unit charge involves a 2π twist of θ as one encircles the defect. More formally,

$$\oint \nabla\theta \cdot d\ell = 2\pi n \quad \Longrightarrow \quad \nabla\theta = \frac{n}{r}\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z,$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_z$ are unit vectors respectively in the plane and perpendicular to it. This (continuum) approximation fails close to the centre (core) of the vortex, where the lattice structure is important.

The energy cost from a single vortex of charge n has contributions from the core region, as well as from the relatively uniform distortions away from the centre. The distinction between regions inside and outside the core is arbitrary, and for simplicity, we shall use a circle of radius a to distinguish the two, i.e.

$$\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a d^2\mathbf{x}(\nabla\theta)^2 = \beta E_n^0(a) + \pi K n^2 \ln\left(\frac{L}{a}\right).$$

The dominant part of the energy comes from the region outside the core and diverges logarithmically with the system size L. The large energy cost associated with the defects prevents their spontaneous formation close to zero temperature. The partition function for a configuration with a single vortex of charge n is

$$\mathcal{Z}_1(n) \approx \left(\frac{L}{a}\right)^2 \exp\left[-\beta E_n^0(a) - \pi K n^2 \ln\left(\frac{L}{a}\right)\right],$$

where the factor of $(L/a)^2$ results from the configurational entropy of possible vortex locations in an area of size L^2. The entropy and energy of a vortex both grow as $\ln L$, and the free energy is dominated by one or the other. At low temperatures, large K, energy dominates and $\mathcal{Z}_1$, a measure of the weight of configurations with a single vortex, vanishes. At high enough temperatures, $K < K_n = 2/(\pi n^2)$, the entropy contribution is large enough to favour spontaneous formation of vortices. On increasing temperature, the first vortices that appear correspond to $n = \pm 1$ at $K_c = 2/\pi$. Beyond this point many vortices appear and the equation above is no longer applicable.

In fact this estimate of K_c represents only a lower bound for the stability of the system towards the condensation of topological defects. This is because pairs (dipoles) of defects may appear at larger couplings. Consider a pair of charges ± 1 separated by a distance d. Distortions far from the core $|\mathbf{r}| \gg d$ can be obtained by superposing those of the individual vortices

$$\nabla\theta = \nabla\theta_+ + \nabla\theta_- \approx 2\mathbf{d} \cdot \nabla \left(\frac{\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z}{|\mathbf{r}|} \right),$$

which decays as $d/|\mathbf{r}|^2$. Integrating this distortion leads to a finite energy, and hence dipoles appear with the appropriate Boltzmann weight at any temperature. The low temperature phase should therefore be visualised as a gas of tightly bound dipoles, their density and size increasing with temperature. The high temperature phase constitutes a plasma of unbound vortices.

(c) Let us begin by defining the path integral for the quantum mechanical time evolution operator. Starting from the time-dependent Schrödinger equation for a single particle system,

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle$$

the time evolution operator is defined by

$$|\Psi(t')\rangle = \hat{U}(t', t) |\Psi(t)\rangle, \qquad \hat{U}(t', t) = \exp\left[-\frac{i}{\hbar} \hat{H}(t' - t) \right].$$

In the real space representation

$$U(x', t'; x, t) = \langle x' | \exp\left[-\frac{i}{\hbar} \hat{H}(t' - t) \right] |x\rangle,$$

According to the Feynman path integral, the quantum evolution operator is expressed as the sum over all trajectories subject to the boundary conditions and weighted by the classical action. In the Hamiltonian formulation,

$$U(x', t'; x, t) = \int Dx(t) \int Dp(t) \exp\left[\frac{i}{\hbar} S(p, x) \right],$$

$$S(p, x) = \int_t^{t'} dt'' \left[p\dot{x} - H(p, x) \right],$$

and in the Lagrangian formulation,

$$U(x', t'; x, t) = \int \bar{D}x(t) \exp\left[\frac{i}{\hbar} S(x) \right],$$

$$S(x) = \int_t^{t'} dt'' \left[\frac{m}{2} \dot{x}^2 - V(x) \right].$$

To establish an analogy with statistical mechanics we have to consider propagation in *imaginary* or *Euclidean time* T. In this way, we obtain

$$U(x', t' = -iT; x, t = 0) = \int Dx(\tau) \exp\left[-\frac{1}{\hbar} S(x) \right],$$

where

$$S(x) = \int_0^T d\tau \left[\frac{m}{2} \left(\frac{\partial x}{\partial \tau} \right)^2 + V(x(\tau)) \right].$$

Interpreting the action as a classical free energy functional, and the path integral as a classical partition function, one has the analogy: the transition amplitude for a quantum particle for the time (-iT) is equal to the classical partition function for a string of length T computed at the value $\beta = 1/\hbar$.

A second analogy follows from the fact that the *quantum* partition function for the particle is given by $\mathcal{Z}_{\rm qu} = \mathrm{Tr}\exp[-\beta H]$ and hence,

$$\mathcal{Z}_{\rm qu} = \int dx \langle x | e^{-\beta H} | x \rangle = \int dx U(x, t' = -i\beta\hbar; x, t = 0).$$

Therefore, in quantum statistical mechanics, the inverse temperature plays the role of an imaginary time. [Additional credit is given for explicit examples in each case, e.g. the one-dimensional Ising model and quantum mechanical tunneling in the double well.]

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38. The Renormalisation Group is based on the assumption that close to the critical point, the singular thermodynamic properties are controlled by fluctuations which take place at the length scale of the correlation length ξ. No other length scale enters the problem. By integrating over fast fluctuations, one can follow how the phenomenological parameters which enter the Hamiltonian flow. At the critical point, the correlation length is infinite and the system has a dilation symmetry. In this case one can deduce that the parameters of the Hamiltonian are fixed. By observing the landscape of parameter flows, one can identify the relevant parameters of the theory and the fixed points.

    (a) The renormalisation group procedure involves three steps. The first step involves integrating over the fast fluctuations of the fields. This step is most easily implemented in Fourier space,

$$\beta H[\mathbf{m}(\mathbf{q})] = \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \frac{1}{2}(t + K\mathbf{q}^2)|m(\mathbf{q})|^2 - h \cdot m(\mathbf{q} = 0)$$

Setting

$$\mathbf{m}(\mathbf{q}) = \begin{cases} \mathbf{m}_<(\mathbf{q}) & |\mathbf{q}| < \Lambda/b \\ \mathbf{m}_>(\mathbf{q}) & \Lambda/b < |\mathbf{q}| < \Lambda \end{cases}$$

and integrating over the fast fluctuations $\mathbf{m}_>(\mathbf{q})$ one obtains the renormalised Hamiltonian

$$\beta H[\mathbf{m}_<(\mathbf{q})] = \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \frac{1}{2}(t + K\mathbf{q}^2)|m_<(\mathbf{q})|^2 - h \cdot m_<(\mathbf{q} = 0).$$

The second step of the renormalisation procedure involves a rescaling of the coordinates to restore the resolution. Setting $\mathbf{q} = \mathbf{q}'/b$

$$\beta H[\mathbf{m}_<(\mathbf{q})] = \int \frac{\mathrm{d}^d\mathbf{q}'}{(2\pi)^d}\frac{b^{-d}}{2}(t + Kb^{-2}\mathbf{q}'^2)|m_<(\mathbf{q})|^2 - h\cdot m_<(\mathbf{q}=0).$$

Finally, the third step in the renormalisation procedure involves changing the contrast setting $\mathbf{m}'(\mathbf{q}') = \mathbf{m}_<(\mathbf{q}')/z$. With this definition, the renormalised Hamiltonian takes the form

$$\beta H'[\mathbf{m}'] = \int \frac{\mathrm{d}^d\mathbf{q}'}{(2\pi)^d}\frac{1}{2}(t' + K'\mathbf{q}'^2)|m'\mathbf{q}')|^2 - h\cdot m'(\mathbf{q}'=0)$$

where

$$\begin{aligned} K' &= Kb^{-d-2}z^2 \\ t' &= tb^{-d}z^2 \\ h' &= hz. \end{aligned}$$

(b) The fixed Hamiltonian is obtained when $t = h = 0$ and $K = K'$ requiring $z^2 = b^{d+2}$. As a result one finds that $t' = tb^{y_t}$ where $y_t = 2$ and $h' = hb^{y_h}$ where $y_h = 1 + d/2$. The corresponding free energy density scales as

$$f(t,h) = -\frac{1}{V}\ln\mathcal{Z} = -\frac{1}{b^dV'}\ln\mathcal{Z}' = b^{-d}f(t',h')$$

i.e. $f(t,h) = b^{-d}f(tb^{y_t}, hb^{y_h})$. Setting $tb^{y_t} = 1$, one finds

$$f(t,h) = t^{d/y_t}f(h/t^{y_h/y_t}) = t^{d/2}f(h/t^{1/2+d/4})$$

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39. (a) Expanding the expression for the area we obtain the partition function

$$\beta H[h] = \beta\sigma A = \beta\sigma\int d^{d-1}\mathbf{x}\left[1 + \frac{1}{2}(\nabla h)^2 + \cdots\right] \simeq A_0 + \frac{\beta\sigma}{2}\int d^{d-1}\mathbf{x}(\nabla h)^2$$

(b) Turning to the Fourier representation (and dropping the constant A_0, the Hamiltonian takes the form

$$\beta H[h] = \frac{\beta\sigma}{2}\int(d\mathbf{q})\mathbf{q}^2|h(\mathbf{q})|^2$$

where $(d\mathbf{q}) \equiv d^{d-1}\mathbf{q}/(2\pi)^{d-1}$. As a consequence of the breaking of the continuous symmetry of h under homogeneous translations the spectrum of low-energy fluctuations vanishes as $\mathbf{q}\to 0$ characteristic of Goldstone modes.

(c) Making use of the correlator

$$\langle h(\mathbf{q}_1)h(\mathbf{q}_2)\rangle = (2\pi)^{d-1}\delta^{d-1}(\mathbf{q}_1+\mathbf{q}_2)\frac{1}{\beta\sigma\mathbf{q}_1^2},$$

we obtain the correlator

$$\begin{aligned}\left\langle [h(\mathbf{x})-h(0)]^2\right\rangle &= \int(d\mathbf{q}_1)\int(d\mathbf{q}_2)\left(e^{i\mathbf{q}_1\cdot\mathbf{x}}-1\right)\left(e^{i\mathbf{q}_2\cdot\mathbf{x}}-1\right)\langle h(\mathbf{q}_1)h(\mathbf{q}_2)\rangle \\ &= \frac{4}{\beta\sigma}\int(d\mathbf{q})\frac{\sin^2(\mathbf{q}\cdot\mathbf{x})}{\mathbf{q}^2}.\end{aligned}$$

By inspection of the integrand, we see that for $d \geq 4$, the integral is dominated by $|\mathbf{q}| \gg 1/|\mathbf{x}|$, and

$$\left\langle [h(\mathbf{x})-h(0)]^2\right\rangle \sim \text{const}.$$

In three dimensions, the integral is logarithmically divergent and

$$\langle [h(\mathbf{x})-h(0)]\rangle \sim \frac{1}{\beta\sigma}\ln|\mathbf{x}|.$$

Finally, in two dimensions, the integral is dominated by small $\mathbf{q}$ and

$$\langle [h(\mathbf{x})-h(0)]\rangle \sim |\mathbf{x}|.$$

This result shows that in dimensions less than 4, a surface constrained only by its tension is unstable due to long-wavelength fluctuations.

(d) Taking into account the quadratic term in the Hamiltonian, the membrance becomes tethered to the position $h = 0$. In this case the field fluctuations acquire a mass t.

$$\left\langle [h(\mathbf{x})-h(0)]^2\right\rangle = \frac{4}{\beta}\int(d\mathbf{q})\frac{\sin^2(\mathbf{q}\cdot\mathbf{x})}{t+\sigma\mathbf{q}^2}.$$

Qualitatively, as a consequence of the mass, the height-height correlation function decays exponentially with separation on length scales in excess of the correlation length $\xi = \sqrt{\sigma/t}$.

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40. The divergence of the correlation length at a second order phase transition suggests that in the vicinity of the transition, the microscopic lengths are irrelevant. The critical behaviour is dominated by fluctuations that are statistically self-similar up to the scale ξ. Self-similarity allows the gradual elimination of the correlated degrees of freedom at length scales $x \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at scale ξ.

(a) In Fourier representation the Hamiltonian takes the diagonal form

$$\beta H = \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d} G^{-1}(\mathbf{q})|m(\mathbf{q})|^2 - hm(\mathbf{q}=0),$$

where the anisotropic propagator is given by

$$G^{-1}(\mathbf{q}) = t + Kq_{\|}^2 + L\mathbf{q}_{\perp}^4.$$

(b) Course-Graining procedure: Separate the field m into components which are slowly and rapidly varying in space.

$$m(\mathbf{q}) = \begin{cases} m_<(\mathbf{q}) & 0 < |q_{\|}| < \Lambda/b \text{ and } 0 < |\mathbf{q}|_{\perp} < \Lambda/c, \\ m_>(\mathbf{q}) & \Lambda/b < |q_{\|}| < \Lambda \text{ or } \Lambda/c < |\mathbf{q}|_{\perp} < \Lambda. \end{cases}$$

In this parameterisation, the Hamiltonian is separable. As such, an integration over the fast degrees of freedom can be performed explicitly.

$$\mathcal{Z} = \mathcal{Z}_> \int Dm_< \exp\left[-\frac{1}{2}\int_0^{\Lambda/b}(dq_{\|})\int_0^{\Lambda/c}(d\mathbf{q}_{\perp})G^{-1}(\mathbf{q})|m_<(\mathbf{q})|^2 + hm_<(0)\right],$$

where the constant $\mathcal{Z}_>$ is obtained from performing the functional integral over the fast degrees of freedom. Applying the rescaling $q'_{\|} = bq_{\|}$, and $\mathbf{q}'_{\perp} = c\mathbf{q}_{\perp}$, the cut-off in the domain of momentum integration is restored. Finally, applying the renormalisation $m'(\mathbf{q}) = m_<(\mathbf{q})/z$ to the Fourier components of the field, we obtain

$$\mathcal{Z} = \mathcal{Z}_> \int Dm'(\mathbf{q}')e^{-(\beta H)'[m'(\mathbf{q}')]},$$

where the renormalised Hamiltonian takes the form

$$(\beta H)' = \frac{1}{2}\int (d\mathbf{q})b^{-1}c^{-(d-1)}z^2\left(t + Kb^{-2}{q'_{\|}}^2 + Lc^{-4}{\mathbf{q}'_{\perp}}^4\right)|m'(\mathbf{q}')|^2 - zhm'(0).$$

From the result, we obtain the renormalisation

$$\begin{cases} t' = tb^{-1}c^{-(d-1)}z^2, \\ K' = Kb^{-3}c^{-(d-1)}z^2, \\ L' = Lb^{-1}c^{-(d+3)}z^2, \\ h' = hz. \end{cases}$$

(c) Choosing parameters $c = b^{1/2}$ and $z = b^{(d+5)/4}$ ensures that $K' = K$ and $L' = L$ and implies the scaling exponents $y_t = 2$, $y_h = (d+5)/4$.

(d) From this result we obtain the renormalisation of the free energy density

$$f_{\text{sing}}(t, h) = b^{-(d+1)/2} f_{\text{sing}}(b^2 t, b^{(d+5)/4}h).$$

Setting $b^2 t = 1$, we can identify the exponents $2 - \alpha = (d+1)/4$ and $\Delta = y_h/y_t = (d+5)/8$.

41. Essay questions

42. (a) Applying the Hubbard-Stratonovich transformation,

$$\exp\left[\sum_{ij} J_{ij}\sigma_i\sigma_j\right] = C\int\prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij}m_j + 2\sum_i \sigma_i m_i\right]$$

the classical partition function $\mathcal{Z} = \sum_{\{\sigma_i\}} e^{-\beta H[\sigma_i]}$ is given by

$$\mathcal{Z} \;=\; C\int\prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij}m_j + \sum_i \ln\left(2\cosh(2m_i+h)\right)\right].$$

(b) To determine $[J^{-1}]_{ij}$, we transform to Fourier space. In particular, for the model at hand, after some algebra, one finds that the eigenvalues of J_{ij} are given by

$$J(q) = \sum_{n=-\infty}^{\infty} e^{iqn} J e^{-\kappa|n|} = \frac{J}{c - b\cos q}$$

where $c = \coth\kappa$ and $b = 1/\sinh\kappa$. Making use of this result we obtain

$$\begin{aligned}
[J^{-1}]_{ij} &= \int_{-\pi}^{\pi} \frac{dq}{2\pi}\frac{e^{-iq(n_i-n_j)}}{J(q)} \\
&= \frac{1}{J}\begin{bmatrix} c & -b/2 & & & \\ -b/2 & c & -b/2 & & \\ & -b/2 & c & -b/2 & \\ & & -b/2 & c & -b/2 \\ & & & -b/2 & c \end{bmatrix}
\end{aligned}$$

Therefore

$$\mathcal{Z} = C\int\prod_k dm_k \exp\left[-\frac{b}{2J}\sum_i (m_i - m_{i+1})^2 - \sum_i U(m_i)\right]$$

where $U(m) = (c-b)m^2/J - \ln[2\cosh(2m+h)]$. In particular $c-b = \tanh(\kappa/2)$.

(c) For small m and h the effective free energy can be expanded as

$$U(m) = -\ln 2 + \frac{t}{2}m^2 + \frac{4}{3}m^4 - 2hm + \cdots$$

where $t/2 = \tanh(\kappa/2)/J - 2$. Evidently, at zero magnetic field, the effective potential $U(m)$ is quartic. For $t < 0$, the potential takes the form of a double well.

The path integral for a particle in a potential well is given by

$$\begin{aligned}\mathcal{Z} &= \int Dr(t)\exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{m}{2}\dot{r}^2 - U(r)\right)\right]\\ &= \int Dr(\tau)\exp\left[-\frac{1}{\hbar}\int_0^T d\tau'\left(\frac{m}{2}\dot{r}^2 + U(r)\right)\right]\end{aligned}$$

By identifying r with m, and τ with x, the partition function of the Ising model is seen to be equivalent to the path integral of a particle in a double well potential where the transition time T is equalent to the length of the spin chain L. The inevitability of quantum mechanical tunneling in the long-time limit is compatible with the absence of long-range order in the one-dimensional magnetic system. Pursuing the analogy, the presence of a magnetic field induces an asymmetry of the potential which localises the majority of the wavefunction in one of the two wells. This corresponds to the appearance of a net magnetisation in the system.

In two-dimensions, the mapping involves the tunneling of an extended string between two quantum well channels. In this case the tunneling is strongly suppressed.
